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ON FUNCTIONAL EQUATIONS CONNECTED WITH DIRECTED DIVERGENCE, INACCURACY AND GENERALIZED DIRECTED DIVERGENCE

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The measures directed divergence, inaccuracy as well as generalized directed divergence occurring in information theory can be characterized by the symmetry, expansibility, branching, and additivity properties together with some regularity and initial conditions. In this paper some functional equations generalizing those implicit in these characterizations shall be treated.

1. **Introduction.** Let $\Delta_n = \{P = (p_1, p_2, \cdots, p_n) \mid p_i \geqq 0$ and $\sum_{i=1}^n p_i = 1\}$ and $\Delta'_n = \{P = (p_1, p_2, \cdots, p_n) \mid p_i > 0$ and $\sum_{i=1}^n p_i \leqq 1\}$ be the set of all finite complete and incomplete probability distributions respectively. In 1948 C. E. Shannon [16] introduced the following measure of information

$$H_n(P) = -\sum_{i=1}^n p_i \log p_i \;, \tag{1.1}$$

on Δ_n which is now known as Shannon's entropy. This has been generalized to inaccuracy [10]. Inaccuracy and the related quantities directed divergence or information gain [11, 15] and generalized directed divergence [3] are given by

$$H_n(P \| Q) = -\sum_{i=1}^n p_i \log q_i \;, \qquad (P \in \Delta_n,\ Q \in \Delta_n \text{ or } \Delta'_n) \;, \tag{1.2}$$

$$I_n(P \| Q) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \;, \qquad (P \in \Delta_n,\ Q \in \Delta_n \text{ or } \Delta'_n) \;, \tag{1.3}$$

and

$$D_n(P \| Q \mid R) = \sum_{i=1}^n p_i \log \frac{q_i}{r_i} \;, \qquad (P \in \Delta_n,\ Q,\ R \in \Delta_n \text{ or } \Delta'_n) \tag{1.4}$$

respectively. While characterizing these measures we come across the following functional equations

$$\sum_{i=1}^n \sum_{j=1}^m F(p_i q_j) = \sum_{i=1}^n F(p_i) + \sum_{j=1}^m F(q_j) \;, \quad (P \in \Delta_n,\ Q \in \Delta_m) \;, \tag{1.5}$$

$$\sum_{i=1}^n \sum_{j=1}^m F(p_i q_j, x_i y_j) = \sum_{i=1}^n F(p_i, x_i) + \sum_{j=1}^m F(q_j, y_j) \;, \tag{1.6}$$

$$(P \in \Delta_n,\ Q \in \Delta_m,\ X \in \Delta_n \text{ or } \Delta'_n,\ Y \in \Delta_m \text{ or } \Delta'_m)$$