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ENGEL LIE RINGS WITH CHAIN CONDITIONS

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A result of Max Zorn states that if a Lie ring satisfies the maximal condition for subrings and if each element is a bounded left Engel element then the Lie ring is nilpotent. The purpose of this paper is to extend this result to Lie rings satisfying the general Engel condition and with no infinite strictly ascending chains of abelian subrings. A similar result was obtained by I. N. Stewart for locally nilpotent Lie algebras.

2. Notation and terminology. Let $\mathfrak{r}$ be a noetherian ring (i.e., commutative associative ring with unit and satisfying the ascending chain condition on ideals). Following Barnes [1, 2] we define a *Lie algebra over* $\mathfrak{r}$ to be an $\mathfrak{r}$-module which is a Lie ring and satisfies for x, y in the Lie ring and $r \in \mathfrak{r}$,

$$r[x, y] = [rx, y] = [x, ry] .$$

(Here [,] denotes Lie multiplication.) Let L be a Lie algebra over $\mathfrak{r}$.

If A is a subset of L we write $A \subseteq L$; if in addition A is an $\mathfrak{r}$-submodule and a Lie subring we write $A \leq L$ and call A a subalgebra of L. In general $\langle A \rangle$ will denote the subalgebra of L generated by A. If $A = \{a\}$ then

$$\langle a \rangle = \mathfrak{r}a = \{ra \mid r \in \mathfrak{r}\} = \langle \{a\} \rangle ,$$

and we call $\langle a \rangle$ a cyclic $\mathfrak{r}$-module. An $\mathfrak{r}$-module A is said to be finite dimensional over $\mathfrak{r}$ if it is a sum of finitely many cyclic $\mathfrak{r}$-modules. If $A = \{a_1, \cdots, a_n\}$ we define $\langle a_1, \cdots, a_n \rangle = \langle A \rangle$.

Let $A, B \subseteq L$. We define $[A, B]$ to be the $\mathfrak{r}$-submodule spanned by the products $[a, b]$ for $a \in A$ and $b \in B$. We also define inductively, $[A, {}_0B] = A$ and $[A, {}_{n+1}B] = [[A, {}_nB], B]$. If $x, y \in L$ then $[x, {}_0y] = x$ and $[x, {}_{n+1}y] = [[x, {}_ny], y]$. An $\mathfrak{r}$-submodule H is said to be an ideal if $[H, L] \subseteq L$; in this case we write $H \lhd L$. If $A \subseteq L$ then

$$I_L(A) = \{x \in L \mid [A, x] \subseteq A\}$$

and

$$C_L(A) = \{x \in L \mid [A, x] = 0\} .$$

If A is an $\mathfrak{r}$-submodule then