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SOME REMARKS ON HARMONIC MEASURE IN SPACE

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The purpose of this paper is to examine the relationship between harmonic measure and $n-1$ dimensional Hausdorff measure for a class of domains in R^n with irregular boundaries. It is shown for these domains that harmonic measure and Hausdorff measure have the same null sets.

This investigation was motivated in part by the work of Hunt and Wheeden, [5], [6]. In these papers they consider Lipschitz domains, that is, domains whose boundaries are locally representable by graphs of Lipschitz functions. One of their main results is that a positive harmonic function defined on a Lipschitz domain has a nontangential limit at all points of the boundary except possibly those that belong to a set of harmonic measure zero. In the classical case where the domain is taken to be the half-space of R^n, the nontangential limit is known to exist at H^{n-1} almost every point of the boundary, c.f., [2], [3]. We will show that for domains Ω satisfying a geometric measure theoretic condition, H^{n-1} (restricted to the boundary of Ω) and harmonic measure have the same null sets. Therefore, for these domains, the results of Hunt and Wheeden will represent a generalization of the classical case.

By use of the conformal mapping theorem it is not difficult to prove, for a domain in R^2 whose boundary is a simple closed rectifiable curve, that harmonic measure and H^1 measure have the same null sets. In § 4 it will be shown that the analog of this does not hold in R^3. We give an example of a topological 2-sphere whose boundary has finite H^2 measure and has a tangent plane at each point, but for which H^2 measure is not absolutely continuous with respect to harmonic measure.

2. Preliminaries. Let Ω be a bounded open subset of R^n and consider the Banach space $C(\partial\Omega)$, the space of continuous functions on the compact set $\partial\Omega$ with the norm given by $\sup\{|f(y)|: y \in \partial\Omega\}$, $f \in C(\partial\Omega)$. For each $x \in \Omega$, let $\lambda_x: C(\partial\Omega) \to R^1$ be the bounded linear functional defined by $\lambda_x(f) = u_f(x)$, where u_f is the harmonic function corresponding to the boundary values f. Hence, there is a unit measure μ_x on $\partial\Omega$ called *harmonic measure*, such that

$$\lambda_x(f) = u_f(x) = \int f d\mu_x\ ,$$

$f \in C(\partial\Omega)$. If G is a component of Ω, the class of Borel subsets of