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A "GOING DOWN" THEOREM FOR CERTAIN REFLECTED RADICALS

B. J. GARDNER AND PATRICK N. STEWART

In a category $\mathscr{K}$ suitable for radical theory, a functor Φ: $\mathscr{K} \to \mathscr{K}$ is studied which is associated with a natural transformation $1_{\mathscr{K}} \to \Phi$ in a way which bears a formal resemblance to the behavior of certain "extension" functors of rings, such as that which assigns to each A the polynomial ring $A[x]$: every normal subobject $N \to \Phi(A)$ has a "contraction" $N^c \to A$. For a radical class $\mathscr{R}$ in $\mathscr{K}$ such that $\mathscr{R}^* = \{A \mid \Phi(A) \in \mathscr{R}\}$ is also radical, some conditions are obtained which imply that $\mathscr{R}^*(A) = \mathscr{R}(\Phi(A))^c$.

1. Preliminaries. We shall work in a category $\mathscr{K}$ for which the general theory of radicals can be developed (for a set of conditions on $\mathscr{K}$ which ensure this and for some other remarks on radicals in categories, see [9]) and shall consider a left-exact functor $\Phi: \mathscr{K} \to \mathscr{K}$ which has associated with it a natural transformation $1_{\mathscr{K}} \to \Phi$, which will be fixed throughout the discussion. We shall further assume that for each normal subobject $N \to \Phi(A)$ there is a normal subobject $N^{cA} \to A$ and a pullback

$$\begin{array}{ccc} N^{cA} & \longrightarrow & A \\ \downarrow & & \downarrow \\ N & \longrightarrow & \Phi(A) \end{array}$$

where the right-hand vertical map is defined by the natural transformation mentioned above. When no confusion can result, N^{cA} will be abbreviated to N^c. We shall frequently find it convenient to write A^e for $\Phi(A)$. A prototypical example of such a functor is that which assigns to each ring A its polynomial ring $A[x]$, in which case $A^e = A[x]$ ("extension") and $N^c = N \cap A$ ("contraction"). The symbol $A \to A^e$ will always denote a map defined by the given natural transformation.

Our category-theoretic terminology is essentially that of [2]. We shall not distinguish notationally between a subobject and a representative map. In particular if $A \in \mathscr{K}$ and $\mathscr{R}$ is a radical class, $\mathscr{R}(A) \to A$ will denote the $\mathscr{R}$-radical of A.

PROPOSITION 1.1.
(a) *If $N \to A$ is a normal subobject, then $N \to A \subseteqq N^{ec} \to A$.*
(b) *If $N_1 \to A^e \subseteqq N_2 \to A^e$ are normal subobjects then $N_1^c \to A \subseteqq N_2^c \to A$.*