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# TOTAL POSITIVITY AND REPRODUCING KERNELS

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**In this paper we investigate the relationship between total positivity and reproducing kernels. We extend the notion of total positivity to domains in the complex plane. In doing so, we also give a geometrical interpretation to certain Wronskians of reproducing kernels. These geometrical quantities are connected to Gaussian curvatures of Kähler metrics induced by these kernels. For simply-connected domains these curvatures are negative constants, thereby showing that the kernels are totally positive and moreover yielding an efficient method for computing the relevant determinants. In general, the reproducing kernels of multiply-connected domains are not totally positive.**

The motivation for this work stems from the work of Karlin [7] which deals with "optimal" quadrature formulas.

Let $H$ be a Hilbert space of functions analytic in a plane domain $D$ and possessing a reproducing kernel $K(z, \bar{t})$, $z, t \in D$. Let $L \in H^*$, where $H^*$ is the dual of $H$. A subset $\mathscr{Q}$ of $H^*$ is specified and a member $Q \in \mathscr{Q}$ is called a quadrature formula. To each $Q \in \mathscr{Q}$ is associated a remainder functional $R_Q = L - Q$. An optimal quadrature formula, if it exists, is any member $Q^* \in \mathscr{Q}$ satisfying

$$||R_{Q^*}|| = \operatorname*{Inf}_{Q \in \mathscr{Q}} ||R_Q|| = \operatorname*{Inf}_{\hat{Q} \in \hat{\mathscr{Q}}} ||\hat{L} - \hat{Q}|| ,$$

where $\hat{L}$ is the representor of $L$ in $H^*$ and $\hat{\mathscr{Q}}$ is the set of all representors of functionals in $\mathscr{Q}$. Since $H$ has a reproducing kernel $K(z, \bar{t})$ it follows (see [5], pp. 318–319) that $\hat{L}(t) = \overline{L_z(z, \bar{t})}$ and

$$||R_{Q^*}||^2 = (L_t - Q_t^*)\overline{(L_z - Q_z^*)K(z, \bar{t})} .$$

(The subscript in $L_z$ indicates that $t$ is held fixed and $L$ is applied to $K(z, \bar{t})$ as a function of $z$.)

More specifically, let $\gamma$ be a rectifiable curve lying in $D$ and specify

$$L(f) = \int_\gamma f(z)w(z)dz , \quad f \in H ,$$

where $w(z)$ is an integrable function on $\gamma$. Consider

$$\mathscr{Q}_n = \{Q \in H^*: Q(f) = \sum_{k=1}^n \alpha_k f(t_k),\ (t_k)_1^n \subset D,\ (\alpha_k)_1^n \subset C\} ,$$