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ON A LEMMA OF BISHOP AND PHELPS

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The main purpose of the present note is to establish a theorem on the existence of maximal elements in certain partially ordered uniform spaces. The theorem unifies a lemma of E. Bishop and R. R. Phelps and a number of known extensions of this lemma.

1. In the proof of the fundamental theorem of E. Bishop and R. R. Phelps [1, Theorem 1] on the density of the set of support points of a closed convex subset of a Banach space, a lemma [1, Lemma 1] on the existence of maximal elements in certain partially ordered complete subsets of a normed linear space played a central role. Subsequently, this lemma was extended for various other purposes. Since none of the extensions is sufficiently general to cover all of them, it seems natural to look for a common generalization. In § 2 we shall present such a general theorem. Actually, we first prove a theorem (Theorem 1) which is too general to be directly applicable, at least in the present context, and next we apply it to obtain the desired theorem (Theorem 2). The theorems are formulated in terms of uniform structures, due to the facts that completeness is the crucial assumption and that both nonmetric topological linear spaces and metric nonlinear spaces are to be covered. The proofs are heavily influenced by known arguments. In § 3 we shall discuss the relations of Theorem 2 to the Bishop-Phelps lemma and its extensions. Finally, in § 4 we shall give a simplified proof of a recent result of J. Daneš by applying the Bishop-Phelps procedure.

2. Everywhere in the following E is assumed to be a nonempty set. By an *extended real valued function* on E we shall mean a mapping $\varphi: E \to]-\infty, +\infty]$, not identically $+\infty$. The set of points $x \in E$ such that $\varphi(x) < +\infty$ is denoted $\operatorname{dom} \varphi$. When E is equipped with a uniformity $\mathscr{U}$, and φ is an extended real valued function on E, we shall say that φ is *inf-complete* when the set of points $x \in E$ such that $\varphi(x) \leqq r$ is complete for each real r. Note that this implies lower semi-continuity of φ, and that the converse holds if $\mathscr{U}$ is a complete uniformity. By an *ordering* on E we shall mean a reflexive, asymmetric and transitive relation $\leqq$; the corresponding irreflexive relation is denoted $\prec$. For $x \in E$ we shall denote by $S(x, \leqq)$ the set of points $y \in E$ such that $x \leqq y$. We shall say that an extended real valued function φ on E is *decreasing* resp.