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THE ORTHOCENTRIC SIMPLEX AS AN EXTREME SIMPLEX

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Let $\mathscr{A}$ be a variable n-simplex containing a fixed point Q and having vertices A_i and corresponding opposite faces $\mathscr{A}_i$, $i = 0, 1, \cdots, n$. We use the properties of orthocentric simplexes to present brief solutions to the following problems and obtain several Erdös-Mordell type inequalities as a by-product, some of which are stronger than known inequalities.

(i) Maximize the volume of $\mathscr{A}$ given the distances $QA_i = d_i \geqq 0$, $i = 0, \cdots, n$.

(ii) Minimize the volume of $\mathscr{A}$ given the distances $e_i \geqq 0$ from Q to $\mathscr{A}_i$, $i = 0, \cdots, n$.

(iii) Find the extrema of (i) and (ii) when only the power means of the distances are given.

(iv) Construct an orthocentric simplex given the lengths of the altitudes.

(v) Maximize the volume of $\mathscr{A}$ given the $(n-1)$-dimensional volumes of the faces.

(vi) Find the maximum in (i) given that Q must be the centroid of $\mathscr{A}$.

(vii) Maximize the volume of the convex hull of a skew $(n+1)$-gon given the power means of its edges.

1. Introduction. Problems (i) and (ii) have been solved recently **[1, 15]**. The present solution is much shorter and shows the relation between the two problems. Problem (iii) is a generalization of **[14]**.

The relation between the three-dimensional version of problems (iv) and (v) was first noticed in 1773 by Lagrange **[11]** and the n-dimensional problems completely solved in 1866 by C. W. Borchardt **[4]**. Unaware of this latter paper, in 1953 M. A. Marmion **[12]** solved the three-dimensional problems anew, obtaining similar results. The present paper obtains Borchardt's results for (iv) and (v) and shows that problem (vi) and problem (vii), which generalizes the result of **[13]**, are also related to these two.

All these solutions depend on the properties of an orthocentric simplex, that is, a simplex whose altitudes A_iH_i, $i = 0, \cdots, n$, concur at its orthocenter. (In general, the altitudes of a simplex are not concurrent but are associated **[8, 9]** a weaker property.) The following facts about an orthocentric simplex $\mathscr{A}$ with orthocenter H are known **[7]** and easily proved. Let $\mathscr{I} = \{0, 1, \cdots, n\}$, $\mathscr{I}' = \{1, \cdots, n\}$ and let the summa-