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AN OBSTRUCTION TO EXTENDING ISOTOPIES OF PIECEWISE LINEAR MANIFOLDS

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Let $F: M \times I^n \to Q \times I^n$ be an n-isotopy (not necessarily PL) of a compact PL m-manifold M in a PL q-manifold Q, and let $G: Q \times I^n \to Q \times I^n$ be an ambient isotopy of Q which covers F on $Q \times \partial I^n$. If $m \leqq q-3$ there is in $\pi_n \mathrm{PL}(M,Q)$ an obstruction to finding an ambient isotopy of Q, isotopic to G, which covers F and agrees with G on $Q \times \partial I^n$.

1. Introduction. In the proof of the Hudson-Zeeman covering isotopy theorem [6], one has no control over the homeomorphism of the ambient manifold which one obtains at the end of the isotopy. In general, one might ask for sufficient conditions under which a given n-isotopy $F: M \times I^n \to Q \times I^n$ of one PL manifold in another, fixed on ∂M, can be covered by an ambient n-isotopy $H: Q \times I^n \to Q \times I^n$ fixed on ∂Q, in such a way that $H \mid Q \times \partial I^n$ is equal to some given level-preserving homeomorphism G of $Q \times \partial I^n$ which covers $F \mid M \times \partial I^n$. Necessary conditions are that F be level-preservingly locally flat and that G have some extension to $Q \times I^n$ which is fixed on ∂Q. That these conditions are not sufficient can be seen by considering an isotopy $F: S^1 \times I \to I^2 \times I$ of a circle in the interior of I^2 which rotates the circle through 360°. Since F can be chosen PL and locally flat, it follows from the ordinary covering isotopy theorem [6] that F can be covered by an ambient isotopy H of I^2 which is fixed on ∂I^2. But if $G: \partial(I^2 \times I) \to \partial(I^2 \times I)$ is the identity homeomorphism, then H cannot be an extension of G. The difficulty here arises from the fact that the space of embeddings of S^1 into I^2 is not simply connected. The theorem below extends results of Gluck, Husch, and Rushing [3,8]. Let M and Q be PL m- and q-manifolds respectively, with M compact, and let $\mathrm{PL}(M,Q;f)$ denote the semi-simplicial complex of proper PL embeddings of M into Q, with base point f.

THEOREM 1. *Let $F: M \times I^n \to Q \times I^n$ be a proper level-preservingly locally flat n-isotopy* (not necessarily PL) *fixed on ∂M. Let $G: Q \times I^n \to Q \times I^n$ be an ambient n-isotopy of Q, fixed on ∂Q, such that $G \circ (F_0 \times 1) \mid M \times \partial I^n = F \mid M \times \partial I^n$. Suppose that $m \leqq q-3$. Then there is a homeomorphism h of Q such that hF_0 is PL and an obstruction α in π_n $\mathrm{PL}(M, Q; hF_0)$ such that $\alpha = 0$ if and only if there is a level-preserving isotopy K of $Q \times I^n$, fixed on $\partial(Q \times I^n)$, such that $K_1 G \circ (F_0 \times 1) = F$;* i.e. *$K_1 G$ extends $G \mid Q \times \partial I^n$ and covers F.*