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A COUNTEREXAMPLE IN THE THEORY OF DEFINABLE AUTOMORPHISMS

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As it is well known, the groups of definable automorphisms of two elementary equivalent relational structures satisfy the same $\forall_1$-statements. We show that this does not hold in general for $\forall_2$-statements, thus correcting an error in the literature.

0. An automorphism φ of a model $\mathfrak{M}$ is said to be definable if there is a formula H of the (first-order) language of $\mathfrak{M}$ and elements $a_1, \cdots, a_n \in M$, such that for all $x, y \in M$

$$\mathfrak{M} \models H(x, y, a_1, \cdots, a_n) \quad \text{iff} \quad \varphi(x) = y.$$

Let Def Aut ($\mathfrak{M}$) denote the group of definable automorphisms of $\mathfrak{M}$ (see [5]).

In [1] it is remarked that if $\mathfrak{M}$ and $\mathfrak{N}$ are elementary equivalent, then Def Aut($\mathfrak{M}$) and Def Aut($\mathfrak{N}$) are universally equivalent. In this note we show that this is the best possible result. We give an example of an $\forall\exists$-statement, which holds in Def Aut($\mathfrak{M}$) but not in Def Aut($\mathfrak{M}'$), where $\mathfrak{M}$ and $\mathfrak{M}'$ are two elementary equivalent models. In fact our $\mathfrak{M}'$ is an elementary extension of $\mathfrak{M}$. This disproves Theorems 1,2 in [3] (p. 109).

We construct our example from the Prüfer group $\mathbf{Z}(3^\infty)$ and investigate definability using the method of Ehrenfeucht games.

1. Our example is as follows. H is the (group theoretical) statement

$$\forall x \exists y \; x = y^2.$$

We define $\mathfrak{M}$ to be $(M, Z, \omega, <, f)$, where M is the disjoint union of Z and ω. Z is the underlying set of the Prüfergroup $\mathbf{Z}(3^\infty)$, which is defined as

$$\left\{ \frac{n}{3^m} \,\middle|\, n, m \in \mathbf{Z} \right\} / \mathbf{Z}.$$

$<$ is the natural ordering of ω, the set of natural numbers. f is a binary function defined by