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A MEASURE OF CONVEXITY FOR COMPACT SETS

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For a subset M of d-dimensional real vector space R^d let

$$c(M) = \inf\{\lambda \geqq 0 | M + \lambda \operatorname{conv} M \text{ is convex}),$$

where $\operatorname{conv} M$ is the convex hull of M and $+$ denotes vector addition of sets. Among the compact subsets of R^d, the convex sets are characterized by the equality $c(M) = 0$. It is proved that $c(M) \leqq d$ for arbitrary subsets of R^d, with equality if and only if M consists of $d + 1$ affinely independent points. If M is either unbounded or connected, then $c(M) \leqq d - 1$; the bound $d - 1$ is best possible in either case.

For subsets M_1, M_2 of d-dimensional real vector space R^d, the Minkowski (or vector) sum is defined by

$$M_1 + M_2 = \{x_1 + x_2 | x_i \in M_i, i = 1, 2\}.$$

Minkowski addition plays an essential role in the theory of convex bodies, due to the fact that the sum of convex sets is always convex. On the other hand, the sum $M_1 + M_2$ may be convex without M_1, M_2 being convex; for instance, if M is the boundary of a convex body K, then $M + M = K + K$. Moreover, it is easy to see that the sum of an arbitrary subset $M \subset R^d$ and a suitable multiple of its convex hull is always convex. This leads us to the definition below. In the following, the abbreviations cl, int, rel int, bd, aff, conv, dim denote, respectively, closure, interior, relative interior, boundary, affine hull, convex hull, dimension.

For a subset $M \subset R^d$, define

$$c(M) = \inf\{\lambda \geqq 0 | M + \lambda \operatorname{conv} M \text{ is convex}\}$$

(here $\lambda A = \{\lambda x | x \in A\}$). The empty set $\varnothing$ is considered as convex, hence $c(\varnothing) = 0$. Clearly $M + \lambda \operatorname{conv} M$ is convex for all $\lambda > c(M)$. If we write

$$M_\lambda = (1 + \lambda)^{-1}(M + \lambda \operatorname{conv} M),$$