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SOME THREE-POINT SUBSET PROPERTIES CONNECTED WITH MENGER'S CHARACTERIZATION OF BOUNDARIES OF PLANE CONVEX SETS

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An elementary characterization of those plane sets which are boundaries of convex sets is given together with other results of the same character.

1. Introduction. A theorem of K. Menger [1], see also F. A. Valentine [3], states that a plane compact set S is the boundary of some (bounded) convex set if and only if S satisfies certain simple conditions expressible in terms of the three-point subsets of S. The main result of the present note is an extension of Menger's theorem to possibly unbounded closed sets (Theorem 3). We shall start by studying subsets of boundaries of convex sets, the main tool being one of Menger's conditions. Our final result (Theorem 4) is an extension of a theorem of Valentine [3], giving a description of those sets satisfying another of Menger's conditions. The results of this note may be related to an unpublished work of W. M. Swan [2].

The assertion of Theorem 3 is illustrated in the figure: A closed set S containing three noncollinear points is the boundary of a convex set if and only if, for any noncollinear points $x, y, z \in S$, firstly, each of the six closed areas with strong hatching (the dotted lines are medians in triangle xyz) contains points from S other than x, y, z, and secondly, the interior of triangle xyz contains no points from S. Instead of the second condition we might have required that the interior of each of the three weakly hatched angles contained no points from S.

2. Terminology. Everything takes place in the plane. The interior, boundary, and convex hull of a set S are denoted by int S, bd S, and conv S, respectively. The closed and open segments with end-points x and y are denoted by $[x, y]$ and $]x, y[$, respectively. If x, y, z are noncollinear points, $L(x, y)$ and $H(x, y; z)$ denote the line through x and y, and the closed half-plane H with $x, y \in \operatorname{bd} H$, $z \in H$, respectively. A *convex curve* is a connected subset of the boundary of a convex set.