PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

A NOTE ON BANACH SPACES OF LIPSCHITZ FUNCTIONS

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This note is divided into two sections. The first establishes some properties of extreme Lipschitz functions that, it is hoped, will lead to satisfactory ways of characterizing them in general. The second section shows how ideas due to Lindenstrauss can be used to establish the existence of Lipschitz spaces that fail to be injective and fail the approximation property.

Introduction. Our notation will follow essentially that of **[6]** and **[11]**. Given a metric space (S, d), $\mathrm{Lip}(S, d)$ denotes the Banach space of bounded real-valued functions on S with norm given by $\|f\| = \max(\|f\|_\infty, \|f\|_d)$, where

$$\|f\|_d = \sup\{|f(s)-f(t)|\,d^{-1}(s,t) \colon s \neq t\} < \infty.$$

The closed subspace of functions f for which $|f(s)-f(t)| = o(d(s,t))$ is denoted by $\mathrm{lip}(S, d)$. If $A \subset S$, $\tilde{A}$ denotes its complement in S, and if $f\colon S \to R$ is a function, M_f denotes $\{s\colon |f(s)| = \|f\|_\infty\}$.

In **[11]**, Roy showed that a function f is an extreme point of the unit ball of $\mathrm{Lip}(S, d)$, with S the unit interval and d the usual metric, if and only if $|f'| = 1$ a.e. on $\tilde{M}_f$ and $\|f\| = \|f\|_\infty = 1$. (See **[10]** for more along these lines.) The purpose of §1 of this note is to discuss some results that we hope will provide clues to possible characterizations of these extreme points for more general metric spaces by presenting two ways in which the bond imposed by the condition "$|f'| = 1$ a.e. on $\tilde{M}_f$" may be broken.

In §2 we observe that if (S, d) is the unit ball of Enflo's space **[4]**, then $\mathrm{Lip}(S, d)$ fails the approximation property.

We also show that $\mathrm{Lip}(S, d)$, where (S, d) is the Hilbert cube, is not injective, ie., is not a $\mathscr{P}_\lambda$ space for any λ (see **[2**, p. 94**]**). This last result uses techniques due to Lindenstrauss **[8]**, and contrasts with the many examples where $\mathrm{Lip}(S, d)$ is isomorphic to the sequence space l_∞. (See the discussion preceding Proposition 2.2.) The results of §2 point out how large the class of spaces $\mathrm{Lip}(S, d)$ is, although they are always dual spaces **[6]** and, for S infinite, always contain a copy of l_∞ **[7]**.

1. We begin by giving a lemma, mainly for the sake of completeness, which is very similar to the technique of Phelps that was used in **[11**, p. 1159**]**.