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SEMIMODULARITY IN THE COMPLETION OF A POSET

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M. D. MacLaren examined semimodularity in the completion by cuts of a lattice L, and showed that if L is semimodular, atomic, and orthocomplemented then $\bar{L}$ is semimodular [Pacific J. Math. 14 (1964)]. We study here semimodularity in an orthomodular poset P and its completion by cuts $\bar{P}$. In particular, we show that if P is semimodular and orthomodular and contains no infinite chains, then $\bar{P}$ is semimodular if and only if $\bar{P}$ is isomorphic to P. Hence, contrary to the result of MacLaren for lattices, semimodularity is never preserved in the completion by cuts of an orthomodular poset with no infinite chains which is not a lattice. More generally, we show that if P is orthomodular, atomic, and orthocomplete, then the covering condition in P is carried over to $\bar{P}$ if and only if P is isomorphic to $\bar{P}$. As a result, MacLaren's theorem cannot be generalized to posets.

We obtain these results by constructing a new, more convenient way of viewing the completion by cuts of a poset. In §4 we use this characterization of the completion by cuts to provide examples which show that if either the condition that P is atomic, or the condition that P is orthocomplete is removed, then the theorem fails; that is P may fail to be a lattice but both P and $\bar{P}$ may satisfy the covering condition.

Let P be any partially ordered set. For each subset $X \subseteq P$ define X^u to be $\{t \in P: t \geqq x \text{ for all } x \in X\}$ and define $X^l = \{t \in P: t \leqq x$ for all $x \in X\}$. Write X^{ul} for $(X^u)^l$. Then the *completion by cuts* of P is the complete lattice $\bar{P} = \{X^{ul}: X \subseteq P, X \neq \emptyset\}$, ordered by set inclusion [5].

It is straightforward to show that if $P \to P: x \to x'$ is an orthocomplement on P, then $*: \bar{P} \to \bar{P}: X^{ul} \to \{x': x \in X\}^l$ is an orthocomplement on $\bar{P}$ [c.f. 4, MacLaren]. P. D. Finch extended this result by providing necessary and sufficient conditions for the conpletion by cuts of an orthocomplemented poset to be orthomodular [2, Proposition 3.2]. Our Theorem 3.6 also shows the relationship between orthomodularity and the covering condition in $\bar{P}$.

2. Definitions. If a and b are elements of a partially ordered set, write $a < b$ to mean that b *covers* a. A lattice L with zero is said to satisfy the *covering condition* if whenever a is an atom of L and and $b \in L$ with $a \wedge b = 0$, then $b < a \vee b$. As a natural generaliza-