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# ON THE EXISTENCE OF STRONG LIFTINGS IN SECOND COUNTABLE TOPOLOGICAL SPACES

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**Let $X$ be a second countable topological space, $\mathfrak{A}$ a $\sigma$-field of subsets of $X$ containing all open sets and $\mu$ a finite positive measure on $\mathfrak{A}$, such that $(X, \mathfrak{A}, \mu)$ is a complete measure space and $\mu(U) > 0$ for every nonempty open $U \subset X$.**

**Then there exists a lifting $\phi : \mathfrak{A} \to \mathfrak{A}$ which satisfies $U \subset \phi(U)$ for every open subset $U \subset X$.**

**Basic notations and definitions.** Throughout this paper $\mathbf{N}$ denotes the nonnegative integers and $\mathbf{R}_+$ the nonnegative real numbers. Moreover

$X$ is a second countable topological space ($\neq \emptyset$),
$\mathfrak{A}$ is a $\sigma$-field of subsets of $X$ containing all open sets,
$\mu : \mathfrak{A} \to \mathbf{R}_+$ is a countable additive measure, satisfying $\mu(U) > 0$ for every nonempty open subset $U \subset X$.

For $A, B \in \mathfrak{A}$ we denote by $A \subsetneqq B$ the fact that $\mu(A \setminus B) = 0$ and write $A \sim B$ if $A \subsetneqq B$ and $B \subsetneqq A$.

A subset $\mathcal{F} \subset \mathfrak{A}$ is called an $\cap$*-system* iff $\emptyset \in \mathcal{F}$, $X \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$.

For an $\cap$-system $\mathcal{F} \subset \mathfrak{A}$ a mapping $\delta : \mathcal{F} \to \mathfrak{A}$ is a *partial* (lower) *density* iff it satisfies the following conditions:

(i) $A \sim B$ implies $\delta(A) = \delta(B)$

(ii) $A \sim \delta(A)$

(iii) $\delta(\emptyset) = \emptyset$ and $\delta(X) = X$

(iv) $\delta(A \cap B) = \delta(A) \cap \delta(B)$ for every $A, B \in \mathcal{F}$.

A mapping $\delta : \mathfrak{A} \to \mathfrak{A}$ with the above properties is called a (lower) *density* and if moreover

(v) $\delta(X \setminus A) = X \setminus \delta(A)$ for every $A \in \mathfrak{A}$

is fulfilled $\delta$ is a *lifting*.

A lifting or density $\delta : \mathfrak{A} \to \mathfrak{A}$ is called *strong* iff in addition

(vi) $U \subset \delta(U)$ for every open $U \subset X$.

Let us restate one of the fundamental properties of partial densities. For an $\cap$-system $\mathcal{F} \subset \mathfrak{A}$, a partial density $\delta : \mathcal{F} \to \mathfrak{A}$, and $A, B \in \mathcal{F}$ with $A \subsetneqq B$ we have $\delta(A) \subset \delta(B)$. In particular $\delta$ preserves inclusions.