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COUNTABLE ORDINALS AND THE ANALYTICAL HIERARCHY, I

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The following results are proved, using the axiom of Projective Determinacy: (i) For $n \geqq 1$, every Π^1_{2n+1} set of countable ordinals contains a Δ^1_{2n+1} ordinal, (ii) For $n \geqq 1$, the set of reals Δ^1_{2n} in an ordinal is equal to the largest countable Σ^1_{2n} set and (iii) Every real is Δ^1_n inside some transitive model of set theory if and only if $n \geqq 4$.

In general we shall use the terminology and notation of **[3]**. In particular letters $i, j, k, \cdots$ are used as variables over $\omega = \{0, 1, 2, \cdots\}$ and $\alpha, \beta, \gamma, \cdots$ as variables over ${}^\omega\omega$ (= the set of *reals*). For a collection of sets of reals Γ, *Determinacy* (Γ) abbreviates the statement that every set in Γ is determined and *Projective Determinacy* (PD) is the axiom that every projective set is determined.

1. An ordinal basis theorem.

A well known boundedness result in recursion theory asserts that if $WO = \{\alpha : \leqq_\alpha \text{ is a wellordering}\}$, where $\leqq_\alpha = \{(m, n) : \alpha(2^m \cdot 3^n) = 0\}$ and $A \subseteq WO$ is Σ^1_1, then $\sup\{|\alpha| : \alpha \in A\} < \delta^1_1$, where for $\alpha \in WO$, $|\alpha| = \text{length}\ (\leqq_\alpha)$ and $\delta^1_n = \sup\{|\alpha| : \alpha \in \Delta^1_n \& \alpha \in WO\}$. We prove below a generalization of this fact to all odd levels of the analytical hierarchy.

THEOREM 1.1. *Assume Projective Determinacy, when $n \geqq 1$. If $A \subseteq WO$ is Σ^1_{2n+1} and $\sup\{|\alpha| : \alpha \in A\} < \aleph_1$, then $\sup\{|\alpha| : \alpha \in A\} < \delta^1_{2n+1}$.*

Proof. For notational simplicity let us take $n = 1$ as a typical case. Thus let $A \subseteq WO$ be Σ^1_3 and assume $\sup\{|\alpha| : \alpha \in A\} < \aleph_1$. Let $B \subseteq {}^\omega\omega$ be Π^1_2 and $f : {}^\omega\omega \to {}^\omega\omega$ recursive such that $f[B] = A$. Consider then the following game: Player I plays β, player II plays γ and II wins iff $\gamma \in WO \& (\beta \in B \to |f(\beta)| \leqq |\gamma|)$. Clearly player II has a winning strategy in this game. But his payoff set is Σ^1_2, so by a result of Moschovakis **[6]** he has a winning strategy τ which is Δ^1_3. Let $T = \{\beta * \tau : \beta \in {}^\omega\omega\}$, where $\beta * \tau$ is the result of II's moves following τ when I plays β. Then $T \subseteq WO$ and T is $\Sigma^1_1(\tau)$, so by the Boundedness Theorem

$$\sup\{|\gamma| : \gamma \in T\} < \delta^1_1(\tau) < \delta^1_3.$$