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RATIONAL VALUED SERIES OF EXPONENTIALS AND DIVISOR FUNCTIONS

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Recently A. Terras established some (soon to be published) relations between the values of Riemann's zeta function at consecutive positive integral argument and values of certain modified Bessel functions. By combining these relations with some previous results concerning the values of $\zeta(s)$ at odd, positive integers (Grosswald–Nachrichten Akad. Wiss. Göttingen, II Math.–Phys. Klasse 1970, pp. 9–13) it follows that certain infinite series of exponentials and divisor functions (somewhat reminiscent of Lambert series) are rational valued.

Specifically, A. Terras proved [**6**] that for complex ρ, for a,b natural integers and with $K_u(z)$ the modified Bessel function (notation of Watson; see [**1**], especially 10.2.15, page 444),

$$\zeta(2\rho)\Gamma(\rho+1)+(1-\rho)\zeta(2\rho-1)\Gamma(1/2)\Gamma(\rho-1/2) \tag{1}$$

$$=2\pi^{\rho}\sum_{a,b\geqq 1}(b/a)^{\rho-1/2}\{2\pi ab(K_{1.5-\rho}(2\pi ab)+K_{0.5+\rho}(2\pi ab))-K_{0.5-\rho}(2\pi ab)\}$$

holds, provided that $\operatorname{Re}\rho>1$. Formula (1) seems related to results of Berndt [**2**], especially his formula (30), but does not seem to follow trivially from it.

If in (1) we take for ρ a natural integer $m>1$, replace the Bessel functions according to classical formulae (see [**1**], p. 444) and perform some routine transformations, (1) is seen to imply

$$\zeta(2m-1)=\frac{(m-2)!}{(2m-2)!}\left\{\frac{(4\pi)^{2m-1}m!|B_{2m}|}{2(2m)!}\right. \tag{2}$$

$$\left.-\sum_{n=1}^{\infty}e^{-2\pi n}\sigma_{-(2m-1)}(n)\sum_{k=0}^{m}\frac{(m+k-2)!(m(m-1)+k(k-1))}{k!(m-k)!}(4\pi n)^{m-k}\right\}.$$

If we equate these representations of $\zeta(2m-1)$ to those established in [**3**], then we obtain some rather curious formulae, that involve the divisor functions $\sigma_k(n)=\Sigma_{d\mid n}d^k$ for odd, negative $k<-1$. The first few of them read