PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 1, 1975

ON SEMIGROUPS IN WHICH $X = XYX = XZX$ IF AND ONLY IF $X = XYZX$

ZENSIRO GOSEKI

A semigroup S will be called *quasi-rectangular* if the set of idempotents of S is non-empty and a rectangular band ideal of S. The theorems of this note prove in part that the following are equivalent. (1) S is a semilattice of semigroups each of which is either idempotent free or quasi-rectangular. (2) Every $\mathcal{J}$-class of S is either idempotent free or a rectangular subband of S. (3) Every $\mathcal{D}$-class of S is either idempotent free or a rectangular subband of S. (4) S is a semigroup in which for any $x, y, z \in S$, $x = xyx = xzx$ if and only if $x = xyzx$.

Recently M. S. Putcha and J. Weissglass ([4]) have given a characterization of a semigroup each of whose $\mathcal{D}$-classes has at most one idempotent. Using results in [4], this note gives also a characterization of a semigroup each of whose $\mathcal{D}$-classes is either idempotent free or consists of a single idempotent. Also, $\mathcal{D}$ may be replaced by $\mathcal{J}$ in the above statement.

Throughout this note S will denote a semigroup and $E(S)$ the set of idempotents of S. Let the set-valued functions I and $\bar{I}$ on S be defined by $I(x, S) = \{e \mid e \in E(S),\ e = exe\}$ and $\bar{I}(x, S) = \{y \mid y \in S,\ y = yxy\}$, respectively. We shall write E, $I(x)$ and $\bar{I}(x)$ for $E(S)$, $I(x, S)$ and $\bar{I}(x, S)$, respectively, when there is no possibility of confusion.

PROPOSITION 1. *The following are equivalent.*

(1) $\bar{I}(x) \cap \bar{I}(y) = \bar{I}(xy)$ *for every* $x, y \in S$.

(2) $I(x) \cap I(y) = I(xy)$ *for every* $x, y \in S$.

In this case we have $\bar{I}(x) = I(x)$ *for every* $x \in S$.

Proof. (1) $\Rightarrow$ (2) follows from $\bar{I}(x) \cap E = I(x)$ for every $x \in S$.

(2) $\Rightarrow$ (1). We will prove that $\bar{I}(x) = I(x)$ for every $x \in S$. Let $a \in \bar{I}(x)$. Then $a = axa$. Hence $ax = (ax)(ax) = (ax)(ax)(ax)$. Thus $ax \in I(ax) = I(a) \cap I(x)$. Hence $ax \in I(a)$, i.e., $ax = (ax)a(ax)$. Hence $axa = (axa)(axa)$, i.e., $a = a^2$. Therefore $a \in \bar{I}(x) \cap E = I(x)$. Thus $\bar{I}(x) \subseteq I(x)$. Clearly $I(x) \subseteq \bar{I}(x)$. Hence $\bar{I}(x) = I(x)$ for every $x \in S$.

PROPOSITION 2. *Let N be the set of elements x of S such that $\bar{I}(x) = \varnothing$. If N is nonempty then N is an ideal of S and idempotent free.*