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ON TWO THEOREMS OF FROBENIUS

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This note contains simple proofs of two classical theorems of Frobenius, on nonnegative matrices. These concern powers of a primitive matrix and the maximal root of a principal submatrix of an irreducible matrix.

The purpose of this note is to give simple and straightforward proofs for two classical theorems of Frobenius [1].

A matrix A is said to be *nonnegative* (*positive*) if all its entries are nonnegative (positive); we write $A \geqq 0$ $(A > 0)$. A nonnegative square matrix is called *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and D are square; otherwise A is *irreducible*.

It was shown by Frobenius [1] that a nonnegative square matrix has a real *maximal root* r such that $r \leqq |\lambda_t|$ for every root λ_t of A and that to r corresponds a nonnegative characteristic vector. Moreover, if A is irreducible, then the maximal root r of A is simple and there is a positive characteristic vector corresponding to it. An irreducible matrix is said to be *primitive* if its maximal root is *strictly* greater than the moduli of the other roots.

We prove the following remarkable two results due to Frobenius (see [1]; also Theorem 8 and Proposition 4, p. 69, in [2]).

THEOREM 1. *If A is primitive then*

$$A^m > 0$$

for some positive integer m.

THEOREM 2. *The maximal root of an irreducible matrix is greater than the maximal root of any of its principal submatrices.*

Proof of Theorem 1. Let A be a primitive matrix with maximal root r. Then the matrix $1/rA$ is primitive as well, its maximal root is 1, and all its other roots have moduli less than 1. Let

$$S^{-1}\left(\frac{1}{r}A\right)S = 1 \dotplus B, \tag{1}$$