PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 2, 1975

ON SUBRINGS OF RINGS WITH INVOLUTION

PJEK-HWEE LEE

We give a systematic account on the relationship between a ring R with involution and its subrings $\bar{S}$ and $\bar{K}$, which are generated by all its symmetric elements or skew elements respectively.

I. Introduction. Let R be a ring with involution * and $\bar{S}$ the subring generated by the set S of all symmetric elements in R. The relationship between R and $\bar{S}$ has been studied by various authors. In [3] Dieudonné showed that if R is a division ring of characteristic not 2, then either $\bar{S} = R$ or $\bar{S} \subseteq Z(R)$, the center of R. Later Herstein [4] extended this result by proving $\bar{S} = R$ for any simple ring R with $\dim_Z R > 4$ and char. $R \neq 2$. The restriction on characteristic was removed by Montgomery [12]. Recently, Lanski [9] proved that if R is prime or semi-prime, so is $\bar{S}$. In §2 of this paper, we show that $\bar{S}$ can inherit a number of ring-theoretic properties such as primitivity, semi-simplicity, absence of nonzero nil ideals etc.. In doing so, a notion called *symmetric subring*, which is a generalization of $\bar{S}$ and its *-homomorphic images, is introduced so that a group of theorems of the same type, including Lanski's results, can be proved via a more or less unified argument. We show also that numerous radicals of $\bar{S}$ are merely the contractions from those of R. As a consequence, we see that R modulo its prime radical behaves much like $\bar{S}$ in many respects.

In §3 we establish a corresponding theory for $\bar{K}$, the subring generated by all skew elements. The only result hitherto known concerning $\bar{K}$ was as follows [4], [12]: If R is simple and $\dim_Z R > 4$, then $\bar{K} = R$. As a matter of fact, the subring $\overline{K^2}$ is more closely related to R than $\bar{K}$ is. We apply the technique developed in §2 to study the relationship between R and $\overline{K^2}$, and then derive some parallel theorems for $\bar{K}$.

II. Symmetric subrings. Our work depends heavily on the notion of *Lie ideals*. By a Lie ideal U of R we mean an additive subgroup which is invariant under all inner derivations of R. That is, $[u, x] = ux - xu \in U$ for all $u \in U$ and $x \in R$. The following lemma concerning Lie ideals will be referred to frequently in the sequel, and it is a combination of some results in [5].

LEMMA 1. *Let R be a semi-prime ring and U a subring and Lie ideal of R. Then U contains the ideal of R which is generated by $[U, U]$. If U is commutative, then $u^2 \in Z$ for all $u \in U$.*