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# BROWNIAN MOTION AND SETS OF MULTIPLICITY

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**$X(t)$ is Brownian motion on the axis $-\infty < t < \infty$, with paths in $R^n$, $n \geqq 2$. $X(t)$ leads to composed mappings $f \circ X$, where $f$ is a real-valued function of class $\Lambda^\alpha(R^n)$, whose gradient never vanishes. To define the class $\Lambda^\alpha(R^n)$, when $\alpha > 1$, take the integer $p$ in the interval $\alpha - 1 \leqq p < \alpha$ and require that $f$ have continuous partial derivatives of orders $1, \cdots, p$ and these fulfill a Lipschitz condition in exponent $\alpha - p$ on each compact set; to specify further that grad $f \neq 0$ throughout $R^n$, write $\Lambda^\alpha_+$. Then a closed set $T$ is a set of "$\Lambda^\alpha$-multiplicity" if every transform $f(T) \subseteq R^1 (f \in \Lambda^\alpha_+)$ is a set of strict multiplicity—an $M_0$-set (see below). Henceforth we define $b = \alpha^{-1}$ and take $S$ to be a closed linear set.**

THEOREM 1. *In order that $X(S)$ be almost surely a set of $\Lambda^\alpha$-multiplicity, it is sufficient that the Hausdorff dimension of $S$ exceed $b$. It is not sufficient that $\dim S = b$.*

An $M_0$-set in $R$ is one carrying a measure $\mu \neq 0$ whose Fourier-Stieltjes transform vanishes at infinity; the theory of $M_0$-sets is propounded in [1, p. 57] and [8, pp. 344, 348, 383] and Hausdorff dimension is treated in [1, II—III]. Theorem 1 reveals a difference between multi-dimensional Brownian motion and the linear process; for linear paths the critical point is dim $S = \frac{1}{2}b$ [5]. Theorem 2 below contains a sharper form of the sufficiency condition.

THEOREM 2. *Let $S$ be a compact set, carrying a probability measure $\mu$ for which*

$$h(u) \equiv \sup \mu(x, x+u) = o(u^b) \cdot |\log u|^{-1}.$$

*Then $X(S)$ is almost surely a set of $\Lambda^\alpha$-multiplicity.*

1. (Proof of Theorem 2) We can assume that $S$ is mapped by $X$ entirely within some fixed ball $B$ in $R^n$ and that all elements $f$ appearing below are bounded in $\Lambda^\alpha$-norm over $B$ (defined in analogy with the norms in Banach spaces of Lipschitz functions). Moreover we can assume that all gradients fulfill an inequality $\|\nabla\| \geqq \delta > 0$ on all of $B$, and even on all of $R^n$.