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DIFFERENTIABILITY CONDITIONS AND BOUNDS ON SINGULAR POINTS

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It is well-known that a normal arc $\mathscr{A}_4$ of cyclic order four in the conformal plane contains at most finitely many singular points and in fact at most eleven. This bound can be reduced to four in the case of a strongly differentiable $\mathscr{A}_4$. Using a characterization of singular points on such arcs this paper shows that strong differentiability is not a necessary condition for this bound. In fact a much weaker condition, viz., the existence of tangent circles, is sufficient to obtain four as the least upper bound.

In [4] a conformal proof is given for the following result. "A normal arc $\mathscr{A}_4$ of cyclic order four contains at most eleven singular points."

That a strongly differentiable ([3], 3.1) $\mathscr{A}_4$ contains at most four singular points can be found in 4.1.4.3 of [1] and in 3.6 of [4].

In §3 it is shown that assuming only Condition I ([3], 1.5) the maximum number of singular points on such arcs and curves is still four and that this is the best possible bound.

In order to obtain this result it is necessary to categorize the different possible types of singular points on such arcs. This characterization is similar to that of O. Haupt and H. Künneth for linearly singular points of arcs of linear order three ([1], 3.2.1).

The definitions and notations used in this paper can be found in [2] and [3]. We include the notations of ordinary and strong conformal differentiability for the reader's convenience. It is obvious that strong differentiability implies ordinary differentiability.

(a) A point p on an arc $\mathscr{A}$ is said to be (conformally) *differentiable* if it satisfies two conditions:

Condition I. For every point $R \doteqdot p$, and for every sequence of points $s \rightarrow p$, $s \in A$, $s \doteqdot p$, there exists a circle C_0 such that $C(p, s, R) \rightarrow C_0$. C_0 is called the *tangent circle* of $\mathscr{A}$ at p through R and is denoted $C(\tau, R)$.

Condition II. If $s \rightarrow p$, $s \doteqdot p$, there exists a circle $C(p)$ such that $C(\tau, s) \rightarrow C(p)$. $C(p)$ is called the *osculating circle* of $\mathscr{A}$ at p.

(b) A point p on an arc $\mathscr{A}$ is said to be (conformally) *strongly differentiable* if it satisfies the following conditions: