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A NOTE ON STARSHAPED SETS

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If S is a compact subset of R^d, it is shown that S is starshaped if and only if S is nonseparating and the intersection of the stars of the (d-2)-extreme points of S is nonempty.

Let $S \subset R^d$. The (d-2)-*extreme points* of S are by definition those points of S such that if $D \subset S$ is a (d-1)-dimensional simplex then $x \notin \operatorname{relint} D$ (the relative interior of D). The totality of (d-2)-extreme points of S is denoted by $E(S)$. For each $y \in S$ we define $S(y)$, the *star of* y by $S(y) = \{z: [y, z] \subset S\}$, where $[y, z]$ denotes the closed line segment from y to z. S is said to be *starshaped* if $\operatorname{Ker} S \neq \emptyset$ where $\operatorname{Ker} S = \{S(y): y \in S\}$. In [2] it is shown that if S is a compact starshaped set in R^d then $\operatorname{Ker} S = \bigcap \{S(y): y \in E(S)\}$. Thus the following question arises: if S is such that $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, under what hypothesis is S starshaped? It is clearly desirable that the hypothesis should be as weak as possible in order to indicate to what extent $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$ implies that S is starshaped. In [3] it is shown that one suitable hypothesis is that S should have the *half-ray property*, that is, for any point x in $R^d \backslash S$ there is a half-line l with vertex x such that $l \cap S = \emptyset$. Now we note that this hypothesis is a rather strong one especially as it is being used to deduce the fact that a certain set is starshaped. Thus one suspects that a much weaker hypothesis might suffice. This suspicion is further strengthened by the example given in [3] to show that, in fact, some hypothesis is necessary. More precisely, the example given is a *separating set* that is, its complement is not connected. The purpose of this note is to prove the following

THEOREM. *If $S \subset R^d$ is a nonseparating compact set and $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, then S is starshaped.*

Proof. Let $z \in \bigcap \{S(y): y \in E(S)\}$. We shall show that for any x in $R^d \backslash S$, $l(x, z) \cap S = \emptyset$ where $l(x, z)$ is the half-line with vertex x which does not contain z but is such that the line containing $l(x, z)$ does contain z. Clearly this suffices to show that S is starshaped.

Choose x_0 in the complement of the convex hull of S, then $l(x_0, z) \cap S = \emptyset$. Now since S is a nonseparating compact set, its complement is a path-connected unbounded open set (see [1, p. 356]). Thus any point in $R^d \backslash S$ can be "joined" to x_0 by a finite polygonal path in $R^d \backslash s$ such that if t is any segment of the path then the line