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NORMAL HYPERSURFACES

JOSEPH BECKER

The purpose of this note is to give a simple analytic proof of a theorem of Oka: If V is a complex analytic hypersurface whose singular locus has codimension at least two, then V is normal. In other words, every weakly holomorphic function is holomorphic.

This result has since been generalized by Abhankar and Thimm to the case when V is an algebraic complete intersection (which is to say that the ideal of functions holomorphic in the ambient space vanishing on V is generated by k functions, where k is the codimension of V in the ambient space).

Actually we prove a slightly stronger result than Oka's.

THEOREM. *Let V be a complex analytic hypersurface, A a complex analytic subset of V with codimension at least 2. Then there is a bounded linear operator $\phi\colon \mathcal{O}(V - A) \to \mathcal{O}(V)$ such that $\phi(f) \mid V - A = f$.*

Proof. Suppose $V \subset \boldsymbol{C}^n$ and the projection $\pi\colon \boldsymbol{C}^n \to \boldsymbol{C}^{n-1}$ to the first $n-1$ co-ordinates gives an r-sheeted branched cover of V in some neighborhood of the origin with branch set B, $B' = \pi(B)$, $A' = \pi(A)$ and $z' = \pi(z)$. Now π induces a homomorphism ${}_{n-1}\mathcal{O} \to {}_{n}\mathcal{O}/I(V) = \mathcal{O}(V)$ making $\mathcal{O}(V)$ into a finitely generated $\mathcal{O}_{n-1}$ module with generators $1, z_n, \cdots, z_n^{r-1}$. Let $P(z', z_n)$ be the minimal degree polynomial for z_n over ${}_{n-1}\mathcal{O}$; for any $f \in \mathcal{O}(V)$ by the Weierstrass division theorem we have $f = QP + R$ where $R \in {}_{n-1}\mathcal{O}[z_n]$ is a holomorphic polynomial of degree $\leqq r-1$. Hence f can be written as $\sum_{i=0}^{r-1} b_i(z') z_n^{r-i-1} \bmod I(V)$. However the $b_i(z')$'s are unique.

For every $z' \notin B'$, let $\alpha_1(z'), \cdots, \alpha_r(z')$ be the values of z_n on the fiber $\pi^{-1}(z)$ and $f_j = f(z', \alpha_j(z'))$ for $j = 1, \cdots, r$. Then

$$f_j = \sum_{i=0}^{r-1} b_i(z') \alpha_j(z')^{r-i-1} .$$

These equations can be viewed as a system of r linear equations in the r unknowns $b_i(z')$ and solved by Cramer's rule:

$$b_i(z') = \frac{\det[1, \alpha_j, \alpha_j^2, \cdots, \alpha_j^{r-i-2}, f_j, \alpha_j^{r-i}, \cdots, \alpha_j^{r-1}]}{\det[1, \alpha_j, \cdots, \alpha_j^{r-1}]}$$

where in both determinants the entries in the jth row are indicated. The denominator is the Vandermonde determinant $\Delta(\alpha_1, \cdots, \alpha_r)$ and