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PROPERTIES OF MARTINGALE-LIKE SEQUENCES

R. JAMES TOMKINS

The purpose of this paper is to define a new type of stochastic sequence and to explore its properties. These new sequences of random variables, called eventual martingales, generalize the concept of a martingale.

Several known results concerning the almost sure limiting behavior of martingales are shown to remain valid for eventual martingales. In addition, eventual martingales are compared with three other martingale-like sequences.

Consider a probability space $(\Omega, \mathscr{F}, P)$. A stochastic sequence $(X_n, \mathscr{F}_n, n \geqq 1)$ will be called an *eventual martingale* if and only if (iff)

(1) $P[E(X_n | \mathscr{F}_{n-1}) \neq X_{n-1}$ infinitely often (i.o.)$] = 0$.

This says, in effect, that, except on an event of probability zero, the martingale property $E(X_n | \mathscr{F}_{n-1}) = X_{n-1}$ holds for all sufficiently large n. In view of the Borel-Cantelli lemma, $(X_n, \mathscr{F}_n, n \geqq 1)$ is an eventual martingale if $\sum_{n=1}^{\infty} P[E(X_n | \mathscr{F}_{n-1}) \neq X_{n-1}] < \infty$; in particular, every martingale is an eventual martingale.

In §2, a decomposition theorem for eventual martingales will be established and used to generalize some known martingale results. Section 3 will explore the relationship among eventual martingales and three other generalizations of martingales.

Assume throughout that $\mathscr{F}_0$ is the trivial sigma-field. Let $I(A)$ denote the indicator function of an event $A \in \mathscr{F}$.

2. A decomposition theorem.

Crucial to the considerations of this section is the following result.

THEOREM 1. *Let $(X_n, \mathscr{F}_n, n \geqq 1)$ be an eventual martingale. Then there exist stochastic sequences $(M_n, \mathscr{F}_n, n \geqq 1)$ and $(Z_n, \mathscr{F}_n, n \geqq 1)$ such that* (i) $X_n = M_n + Z_n$ *for all* $n \geqq 1$, (ii) $(M_n, \mathscr{F}_n, n \geqq 1)$ *is a martingale, and* (iii) $P[Z_{n+1} \neq Z_n$ i.o.$] = 0$.

Proof. Let $d_1 = X_1$ and, for $n \geqq 1$, let $d_{n+1} = X_{n+1} - X_n$. If $n \geqq 1$, let $M_n = \sum_{k=1}^{n} d_k I(E(d_k | \mathscr{F}_{k-1}) = 0)$ and $Z_n = X_n - M_n$. Then (i) and (ii) are obvious. Moreover, $Z_{n+1} - Z_n = d_{n+1} I(E(d_{n+1} | \mathscr{F}_n) \neq 0)$ so $[Z_{n+1} \neq Z_n] \subseteqq [E(d_{n+1} | \mathscr{F}_n) \neq 0]$. Hence $0 \leqq P[Z_{n+1} \neq Z_n$ i.o.$] \leqq P[E(d_{n+1} | \mathscr{F}_n) \neq 0$ i.o.$] = 0$ by (1).

REMARK. Let $B = [Z_{n+1} \neq Z_n$ i.o.$]$. Theorem 1 (iii) says that,