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REPRESENTATIONS BY SPINOR GENERA

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If f and g are two nonsingular quadratic forms with rational integral coefficients such that f represents g integrally over every p-adic fields and also over the reals, then it is a well-known classical result that the genus $\mathrm{Gen}(f)$ of f represents g. This paper considers the question of how many spinor genera in the genus of f will represent g, when f and g are integral forms defined over some fixed domain of algebraic integers and when $\dim(f) - \dim(g) \geqq 2$.

Unless otherwise mentioned the following general assumptions will be understood throughout this paper: F is an algebraic number field with R as its ring of algebraic integers, V and W are finite dimensional regular quadratic spaces over F with $\dim V - \dim W = d \geqq 2$, L and K are respectively R-lattices on V and W, and S is the set of all discrete spots on F. All unexplained notations and terminologies are from [6]. Suppose now that $L_{\mathfrak{p}}$ represents $K_{\mathfrak{p}}$ for every $\mathfrak{p} \in S$, then it is a well-known result that there is a lattice L' in the genus of L that represents K, provided V represents W (in fact, if W were a subspace of V, this L' may be chosen so as to contain K; see 102:5, [6]). We introduce the notations $K \rightarrowtail \mathrm{Gen}(L)$, $\mathrm{Spn}(L)$, $\mathrm{Spn}^+(L)$, $\mathrm{Cls}(L)$, $\mathrm{Cls}^+(L)$ to denote respectively that K is representable by a member in the genus, spinor genus, proper spinor genus, class, proper class of L. Thus, in this notation, $L_{\mathfrak{p}}$ represents $K_{\mathfrak{p}}$ locally everywhere at $\mathfrak{p} \in S$ and $W \rightarrowtail V$ is equivalent to $K \rightarrowtail \mathrm{Gen}(L)$, which is, of course, the same as representation by $\mathrm{Gen}^+(L)$. We show here that if $d \geqq 3$ then $K \rightarrowtail \mathrm{Gen}(L)$ implies $K \rightarrowtail \mathrm{Spn}^+(L)$ so that in the indefinite case for L every proper class in the genus represents K. This fact must surely have been known to the specialists although I have not seen it in print and choose to record it here for completeness; its proof is quite standard and does not employ any of the subtler or deeper aspects of the theory. On the other hand, when $d = 2$, the theory is a good deal more intricate. We show that here too in most cases K is representable by every proper spinor genus in the genus of L; the exceptional cases will be pointed out, and there one needs to know the precise results for the local computations of the spinor norms of local integral rotations on $L_{\mathfrak{p}}$; the known facts about these are found in [3] for nondyadic $\mathfrak{p}$, in [1] for unramified dyadic $\mathfrak{p}$, and in [2] for arbitrary dyadic $\mathfrak{p}$ but with $L_{\mathfrak{p}}$ modular. This study was motivated by Kneser's paper [4], and the results as well as the method follow closely along his