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HOMOTOPIES AND INTERSECTION SEQUENCES

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For $\gamma_t: S^1 \to \mathbf{C}$, a smooth homotopy of closed curves, the changing configuration of vertices and cusps is studied by considering the set in $I \times S^1 \times S^1$ given by $(\gamma_t(z) - \gamma_t(\zeta))/(z - \zeta) = 0$. The main tool is oriented intersection theory from differential topology. The results relate to previous work by Whitney and Titus on normal curves and intersection sequences.

Consider a closed curve as a smooth map $\gamma: S^1 \to \mathbf{C}$. Let γ_t for $t \in I$ be a smooth homotopy of closed curves. A vertex of γ_t is a point w such that $w = \gamma_t(z) = \gamma_t(\zeta)$ for $z \neq \zeta$. A cusp is a point where the tangent vanishes and changes direction. Let $X = I \times S^1 \times S^1$. We study the changing configuration of vertices and cusps of γ_t by studying the set $Z = \{x \in X \mid G(x) = 0\}$ where $G(t, z, \zeta) = (\gamma_t(z) - \gamma_t(\zeta))/(z - \zeta)$, and the limiting value is taken when $z = \zeta$. If 0 is a regular value for G, then Z has the structure of an oriented 1-submanifold of X. If for fixed t, Z intersects $t \times S^1 \times S^1$ transversely, then the oriented intersection gives a set of pairs in $S^1 \times S^1$ with corresponding orientation numbers $+1$ or -1. If γ_t is a normal immersion, these pairs and their orientation numbers give the Titus intersection sequence of γ_t. The changes in the intersection sequence are reflected in the behavior of Z. If Z crosses $I \times \Delta$, where Δ is the diagonal of $S^1 \times S^1$, then we have a cusp and a change in the tangent winding number. The difference between the tangent winding numbers of γ_0 and γ_1 is just $N(Z, I \times \Delta)$, the total number of oriented intersections of Z with $I \times \Delta$.

1. Intersection sequences. In the complex plane, let S^1 be the set $|z| = 1$. Consider S^1 as a 1-manifold with functions $\theta \to e^{i\theta}$ giving local coordinate systems. The tangent vector $d/d\theta$ is defined independently of the choice of coordinate system. On $T(S^1)$, the tangent space, let $d/d\theta$ give the positive orientation at each point. This gives S^1 the structure of an oriented 1-manifold.

Suppose $\gamma: S^1 \to \mathbf{C}$ is a smooth (C^∞) map. Let $\beta(z) = (d\gamma/d\theta)(z)$ be the tangent at $\gamma(z)$. Let $S^1 \times S^1 = Y$ and let the maps $(\theta, \phi) \to (e^{i\theta}, e^{i\phi})$ give local coordinate systems for Y. Let $S^1 \times S^1$ have the product orientation, i.e., $T(S^1 \times S^1)$ has positive orientation given by the ordered basis $\{\partial/\partial\theta, \partial/\partial\phi\}$ at each point. Let $\Delta \subseteq Y = \{(z, \zeta) \mid z = \zeta\}$.

Let $\theta \to (e^{i\theta}, e^{i\theta})$ be local coordinate systems on Δ and let positive