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A HAHN DECOMPOSITION FOR LINEAR MAPS

RICHARD I. LOEBL

The question is studied whether every bounded self-adjoint linear map φ be tween two C^*-algebras can be written as the difference of bounded positive linear maps. Such a decomposition is called a *Hahn decomposition* of φ.

THEOREM. *Let X be an infinite compact Hausdorff space. Then there is a bounded, self-adjoint linear map, with domain $C(X)$, that does not admit a Hahn decomposition.*

A bounded linear map φ is said to have finite total variation if

$$\sup\left\{\left\|\sum_{i=1}^{n}|\varphi(a_i)|\right\|: a_i \in \mathscr{A}, 0 \leqq a_i, \sum a_i \leqq 1\right\} < \infty .$$

THEOREM. *If the domain is commutative, and if the range is a von Neumann algebra, then a sufficient condition for a self-adjoint map to admit a Hahn decomposition is that the map have finite total variation.*

0. Introduction. It is a well-known theorem [4] that every linear functional τ on a C^*-algebra $\mathscr{A}$ can be written $\tau = \tau_1 - \tau_2 + i(\tau_3 - \tau_4)$, where the τ_j are positive linear functionals. It is, therefore, natural to ask whether every bounded linear map φ between two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ admits a decomposition $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$, where the φ_j are positive linear maps.

Given any bounded linear map φ, if we define the linear map $\tilde{\varphi}$ by $\tilde{\varphi}(a) = \varphi(a^*)^*$, it is easy to see that $\|\tilde{\varphi}\| = \|\varphi\|$, and that $\tilde{\varphi}$ is the natural "adjoint" map to φ. Hence, the map $\varphi_1 = (\varphi + \tilde{\varphi})/2$ is self-adjoint, i.e., $\varphi_1(a^*) = \varphi_1(a)^*$, as is $\varphi_2 = (\varphi - \tilde{\varphi})/2i$, and therefore φ can be written (uniquely) as $\varphi = \varphi_1 + i\varphi_2$, the usual combination of self-adjoint elements.

We are now reduced to the following problem: Given a bounded, self-adjoint linear map φ between two C^*-algebras, when can we write $\varphi = \varphi_1 - \varphi_2$ where φ_1, φ_2 are bounded, positive linear maps?

DEFINITION 0.1. We shall call such a form a *Hahn decomposition* of φ.

In general, a Hahn decomposition is not always possible. Even in the commutative case, pathology can occur [see Theorem 2.2 below].

For future references, we state here Grothendieck's result for