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ON FACETS WITH NON-ARBITRARY SHAPES

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It is proved that the shape of a facet of a d-polytope with $d+3$ vertices can be arbitrarily preassigned. A minimal example of a 4-polytope with 8 vertices which does not have this property is described.

1. Introduction. The shape of a facet F of a polytope P is said to be *arbitrarily preassignable* if, given any polytope F' combinatorially equivalent to F there is a polytope P' combinatorially equivalent to P such that F' is a facet of P' and F' is the image of F under the combinatorial isomorphism which maps P onto P'.

In [3] Barnette and Grünbaum proved that the shape of one n-gonal 2-face F of a 3-polytope can be any preassigned convex n-gon F'. They ask to what extent their result holds in higher dimensions. They mention that there is an 8-polytope P with 12 vertices such that the shape of one of its 7-dimensional faces can not be arbitrarily chosen, and they conjecture that a similar example can be found already in four dimensions.

In [4], such a 4-polytope with 13 vertices is described. We shall describe a smaller example of this type in the proof of our first theorem:

THEOREM. *There is a 4-polytope with 8 vertices such that the shape of one of its 3-faces can not be arbitrarily preassigned.*

From the results in [3] and the following lemma we know that the above theorem yields a minimal example of such a polytope.

LEMMA. *Let P be a d-polytope with $d+3$ vertices. Then the shape of any facet of P can be arbitrarily preassigned.*

Proof of the theorem. We shall prove the theorem by describing a 4-polytope P, the facets of which are given by their vertices in Table 1.

P possesses 15 facets, 14 of them being tetrahedra and one an octahedron. The vertices of the octahedron are labelled like it is described in Figure 1.

First of all, we have to show that the complex described in Table 1 is isomorphic to the boundary-complex of a convex polytope. Those 3-polytopes given in Table 1 which do not contain the vertex 1, are either an octahedron (235678) or the convex hull of the vertex