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INTEGRAL BASES FOR BICYCLIC BIQUADRATIC FIELDS OVER QUADRATIC SUBFIELDS

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Explicit conditions are given for a bicyclic biquadratic number field to have an integral basis over a quadratic subfield.

A classical question of algebraic number theory is, "When does an algebraic number field K have an integral basis over a subfield k?"

A complete and explicit answer to the above question is given here when K is a bicyclic biquadratic number field and k is a quadratic subfield. Moreover, an explicit integral basis is given for K/k whenever one exists. In the cases where k is imaginary or k is real and has a unit of norm -1, the conditions involve only rational congruences. When k is real and the fundamental unit of ϵ has norm $+1$, the conditions sometimes involve ϵ.

1. Notation and preliminary remarks.

Throughout this article the following notation shall be used:
Q: field of rational numbers.
Z: rational integers.
m, n: square free integers.
$l = (m, n) > 0$, $m = m_1 l$, $n = n_1 l$ and $d = m_1 n_1$.
$K = Q(\sqrt{m}, \sqrt{n})$: bicyclic biquadratic field.
$k = Q(\sqrt{m})$.
$\delta_{L/M}$: different of an extension L/M.
$N(\epsilon)$: norm of the unit ϵ.
p, q: odd prime numbers.

An integral basis for K over Q has been determined in **[1, 3, 6]**. Here an integral basis for K over $k = Q(\sqrt{m})$ will be determined whenever it exists. In these considerations the roles of n and d are interchangeable so it will only be necessary to consider seven pairs of congruence classes for (m, n) modulo 4; namely $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 3)$, $(3, 1)$ and $(3, 2)$.

It follows immediately from **[5]** that K has an integral basis over k if and only if $K = k(D^{\frac{1}{2}})$ where (D) is the discriminant of K over k. Since K is a quadratic extension of k the discriminant is the square of the different δ. In **[3, 6]** the different of K over Q is explicitly determined by: