PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No. 2, 1976

EMBEDDINGS OF COMPACT CONVEX SETS AND LOCALLY COMPACT CONES

JIMMIE D. LAWSON

The main result of this paper is that a compact convex set with a basis of neighborhoods (not necessarily open) at each point which are convex can be embedded in a locally convex separated topological vector space. An analogous result is proved for locally compact cones. Along the way it is shown that any compact convex set can be embedded as a base of a locally compact cone in a separated topological vector space, and that the various notions of local convexity coincide in a compact convex set.

1. Introduction. The outstanding open problem in the area with which this paper is concerned has been whether compact convex subsets of general linear spaces can differ from the compact convex subspaces of locally convex spaces or some mild variant thereof. For example, the following is a question posed by V. Klee [8]: Does every element of a compact convex subset K of a topological vector space possess a basis of neighborhoods whose intersection with K is convex?

The main result of this paper provides an affirmative answer for the case that the compact convex set K possesses at each point a basis of neighborhoods (not necessarily open) which are convex. After the original manuscript of this paper had been written, the author learned that J. W. Roberts had obtained the same result some five or six months earlier, and has submitted it for publication [11]. Roberts has also discovered an example of a compact convex set without extreme points, hence which cannot be embedded in a locally convex space [12]. However, the technique of proof which is employed here is quite different. We use these results to sharpen some earlier results on embedding of cones [9].

The main result is a mild improvement on the work of R. E. Jamison, R. C. O'Brien, and P. D. Taylor [5]. In [5], they proved that if a compact, convex set has a base of *open* convex subsets, then it is affinely homeomorphic to a convex subset of a locally convex topological vector space.

The results for compact convex sets have analogues for locally compact cones; these results are also included. Indeed, the motivation for this paper was a further study of locally compact cones, which arise in a natural way in the theory of compact semigroups [6].