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ON CERTAIN ALGEBRAIC INTEGERS AND APPROXIMATION BY RATIONAL FUNCTIONS WITH INTEGRAL COEFFICIENTS

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Let A be a finite set of integers $\{a_1, a_2, \cdots, a_l\}$ and (possibly) ∞. Let X be a nonempty closed subset of $C \cup \{\infty\}$, the field of complex numbers together with ∞, under the topology of the Riemann sphere. Suppose that X is symmetric with respect to the field of real numbers R(i.e. if $z \in X$ then $\bar{z} \in X$) and disjoint from A. We are interested in the following two problems:

I. Under what conditions do there exist, for each neighborhood N of X, infinitely many algebraic numbers θ such that $1/(\theta-a_1), 1/(\theta-a_2), \cdots, 1/(\theta-a_l)$ are algebraic integers and, if $\infty \in A, \theta$ is itself an algebraic integer, such that all of the (algebraic) conjugates of θ lie in N?

II. If X has empty interior and connected complement, then the polynomials are dense in the ring of continuous functions of X. What is the uniform closure of the polynomials with integral coefficients in $1/(x-a_1)$, $1/(x-a_2), \cdots$, $1/(x-a_l)$, and if $\infty \in A, x$ itself?

Problem I was investigated by Raphael Robinson [10]; however instead of requiring the $1/(\theta - a_i)$ to be algebraic integers, he required that the $b_i/(\theta - a_i)$ be algebraic integers, where the b_i are integers satisfying $(a_i - a_j) | b_i$ for each $j \neq i$. Our methods are similar to those of Robinson; there are, however, significant differences.

Throughout the remainder of this paper, A will denote a nonempty finite set consisting of real numbers $a_1, a_2, \cdots, a_l$ and (possibly) ∞. We assume that $|a_i - a_j| \geqq 1$ if $i \neq j$. In §§ 2, 3, 4, we shall assume that the a_i are integers. If $\infty \in A$, we shall sometimes denote it by a_0. By a *symmetric closed* (SC) A-set X, we shall mean a nonempty closed subset of the Riemann sphere, symmetric with respect to the x-axis, satisfying $A \cap X = \emptyset$.

If $P(z)$ is a polynomial, we shall denote the leading coefficient of $P(z)$ by $P(\infty)$.

1. Classification of SC A-sets.

A rational function with real coefficients $\varphi(z)$ is said to be an *A-function* if it is regular except possibly for poles at $a_i \in A$. Such a function can be written uniquely in the form $P(z)/D(z)$ where $P(z)$ is a polynomial, $D(z) = \prod_{i=1}^{l}(z-a_i)^{r_i}$ where the $r_i \geqq 0$ and $P(a_i) \neq 0$ when $r_i > 0$, for $1 \leqq i \leqq l$. If