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COMPOSITION PROPERTIES OF PROJECTIVE HOMOTOPY CLASSES

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1. Introduction. **A homotopy class $x \in \pi_q(X)$ is said to be** *projective on X*, **or** *projectively carried by X*, **if it can be represented by a map that factors through the projective space p^q, as shown in diagram (I), where $\hat{\pi}$ is the double covering map.**

$$
\text{(I)}\quad
\begin{tikzcd}
S^q \arrow[rr, "x"] \arrow[dr, "\hat{\pi}"'] & & X \\
& P^q \arrow[ur, dashed, "f"'] &
\end{tikzcd}
\qquad\qquad
\text{(II)}\quad
\begin{tikzcd}
S^q \arrow[rr, "x"] \arrow[dr, "\pi"'] & & S^m \\
& P^q_t \arrow[ur, dashed] &
\end{tikzcd}
\qquad (t \leqq m).
$$

When x is a stable homotopy class of spheres, it is of interest to ask for the values of m such that x be projective on S^m. Since S^m is $(t-1)$-connected for $t \leqq m$, this amounts to the factorisation problem posed in diagram (II) above, where π is $\hat{\pi}$ followed by the collapsing map from P^q to the truncated projective space $P^q_t = P^q / P^{t-1}$. We give an answer to this problem when x is a generator of the image of the J-homomorphism.

In recent years the problem of projective classes has been studied in **[3]**, **[8]**, **[10]**, **[13]**, and **[14]**. We refer especially to **[8]**, in which the authors determine, for classes up to the 28-stem, the various spheres that can carry them projectively. We also refer to **[10]**, where the authors prove among other things, that every stable homotopy class of spheres is carried projectively by some sphere.

Let u, v be classes in stable stems of spheres, and suppose x is projectively carried by X. For specific u and v, it sometimes turns out that the composition xu, or a representative x' of the secondary composition $\langle x, u, v\rangle$ (i.e., Toda bracket), will also be carried by X projectively. In this paper we prove the following typical results.

THEOREM 5.1. *Let $x \in \pi_q(X)$ be projective on the $(m-1)$-connected space X, where $q \leqq 2m-4$. Then $x\eta$ is projective on X.*

THEOREM 6.2. *Let $x \in \pi_q(X)$ be projective on the $(m-1)$-connected space X, where $q \leqq 2m-10$ and $q \equiv 3 \pmod 4$. If the Toda bracket $\langle x, 8\sigma, 2\rangle$ is defined, then all its representatives are projective on X.*

As an application we deduce Mahowald's Theorem D [7, p. 4] on the vanishing of certain Whitehead products, namely