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ON THE DISTRIBUTION OF a-POINTS OF A STRONGLY ANNULAR FUNCTION

AKIO OSADA

This paper gives an example of a strongly annular function which omits 0 near an arc I on the unit circle C and which omits 1 near the complementary arc $C-I$. This example affirmatively answers the following question of Bonar: Does there exist any annular function for which we can find two or more complex numbers w such that the limiting set of its w-points does not cover C?

1. Introduction. The purpose of this paper is to study the distribution of a-points of annular functions. We recall that a holomorphic function in the open unit disk $D : |z| < 1$ is said to be annular [1] if there is a sequence $\{J_n\}$ of closed Jordan curves about the origin in D, converging out to the unit circle $C : |z| = 1$, such that the minimum modulus of $f(z)$ on J_n increases to infinity as n increases. When the J_n can be taken as circles concentric with C, $f(z)$ will be called strongly annular. Given a finite complex number a, the minimum modulus principle guarantees that every annular function f has infinitely many a-points in D and hence their limit points form a nonempty closed subset, say $Z'(f, a)$, of C. On the other hand, by virtue of the Koebe–Gross theorem concerning meromorphic functions omitting three points, it follows from the annularity of f that open sets $C - Z'(f, a)$ and $C - Z'(f, b)$ on the circle can not overlap if $a \neq b$ and consequently that the set of all values a for which $Z'(f, a) \neq C$ must be at most countable. Therefore we may well say such a to be singular for f.

For this reason we will be concerned with the set $S(f) = \{a : Z'(f, a) \neq C\}$ in this paper. We denote by $|S(f)|$ the cardinality of $S(f)$ and then, from the simple fact observed above, we have that $0 \leqq |S(f)| \leqq \aleph_0$, which in turn conversely tempt us to raise the following question: Given a cardinality $N (0 \leqq N \leqq \aleph_0)$, can we find any annular function f for which $|S(f)| = N$? ([1], [2]).

We know many examples of strongly annular functions such that $|S(f)| = 0$ [4]. In particular if an annular function f belongs to the MacLane class, i.e., the family of all nonconstant holomorphic functions in D which have asymptotic values at each point of everywhere dense subsets of C, the set $S(f)$ becomes necessarily empty. As for $N = 1$, Barth and Schneider [3] constructed an example of an annular function f for which $|S(f)| = 1$. The example involved in their construction,