PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 1, 1977

ON A CLASS OF UNBOUNDED OPERATOR ALGEBRAS III

ATSUSHI INOUE

In this paper we continue our study of unbounded operator algebras begun in previous papers. In particular, the unbounded Hilbert algebras are studied. The primary purpose of this paper is to give necessary and sufficient conditions under which an unbounded Hilbert algebra is pure.

1. Introduction. In the previous paper [6] we began our study of unbounded Hilbert algebras and raised the following problem.

Problem. Let $\mathscr{D}_0$ be a maximal Hilbert algebra in a Hilbert space $\mathfrak{H}$. Does there exist a pure unbounded Hilbert algebra over $\mathscr{D}_0$ in $\mathfrak{H}$?

In this paper we find that if $\mathscr{D}_0 \neq \mathfrak{H}$ then the answer is affirmative. That is, if $\mathscr{D}_0 \neq \mathfrak{H}$, then the maximal unbounded Hilbert algebra $L_2^{\omega}(\mathscr{D}_0)$ is a pure unbounded Hilbert algebra over $\mathscr{D}_0$ in $\mathfrak{H}$. It therefore seems that our study of a class of unbounded operator algebras called $EW^{\#}$-algebras is significant. For, from ([6] Theorem 3.10) if $\mathscr{D}_0 \neq \mathfrak{H}$ then there necessarily exist pure $EW^{\#}$-algebras over the left von Neumann algebra $\mathscr{U}_0(\mathscr{D}_0)$ of $\mathscr{D}_0$ and if $\mathfrak{A}_0$ is a semifinite von Neumann algebra with a faithful normal semifinite trace φ_0 on $\mathfrak{A}_0^+$ and $L^2(\varphi_0) \neq \mathfrak{A}_0 \cap L^2(\varphi_0)$, then there exist pure EW^*-algebras over $\mathfrak{A}_0$ such that are isomorphic to standard $EW^{\#}$-algebras.

2. Basic theory for unbounded Hilbert algebras. We give here only the basic definitions and facts needed. For a more complete discussion of the basic properties of unbounded Hilbert algebras the reader is referred to [6, 7].

Let $\mathscr{D}$ be a pre-Hilbert space with an inner product $(\,|\,)$ and be a $*$-algebra. Let $\mathfrak{H}$ be the completion of $\mathscr{D}$. Suppose that $\mathscr{D}$ satisfies;

$$(\xi\,|\,\eta) = (\eta^*\,|\,\xi^*)\,, \quad \xi, \eta \in \mathscr{D}\,, \tag{1}$$

$$(\xi\eta\,|\,\zeta) = (\eta\,|\,\xi^*\zeta)\,, \quad \xi, \eta, \zeta \in \mathscr{D}\,. \tag{2}$$

Now, we define $\pi(\xi)$ and $\pi'(\xi)$ by;

$$\pi(\xi)\eta = \xi\eta \quad \text{and} \quad \pi'(\xi)\eta = \eta\xi\,, \quad \eta \in \mathscr{D}\,.$$

Then, by (2), we know that $\pi(\xi)$ and $\pi'(\xi)$ are closable operators on $\mathfrak{H}$ with the domain $\mathscr{D}$ and $\pi(\xi)^* \supset \pi(\xi^*)$, $\pi'(\xi)^* \supset \pi'(\xi^*)$.