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INNER-OUTER FACTORIZATION OF FUNCTIONS WHOSE FOURIER SERIES VANISH OFF A SEMIGROUP

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Let G be a compact, connected, Abelian group. Its dual, Γ, is discrete and can be ordered. Let Γ_1 be a semigroup which is a subset of the positive elements for some ordering, but which contains the origin of Γ. Let $H^p(\Gamma_1)$ be the subspace of $L^p(G)$ consisting of functions which have vanishing off Γ_1. The question that this paper is concerned with is what conditions on a function in $H^p(\Gamma_1)$ assure an inner-outer factorization.

An inner function is a function $f \in H^\infty(\Gamma_1)$ such that $|f|=1$ a.e. (dx) on G. A function $f \in H^p(\Gamma_1)$ is said to be outer if

$$\int_G \log |f(x)| = \log \left| \int_G f(x)dx \right| > -\infty .$$

A function $f \in H^1(\Gamma_1)$ is said to be in the class $LRP(\Gamma_1)$ if $\log |f| \in \Gamma_1(G)$ and $\log |f|$ has Fourier coefficients equal to zero off $\Gamma_1 \cup -\Gamma_1$. The main result of the paper is that if Γ_1 is the intersection of half planes and $f \in H^1(\Gamma_1)$ with $\int_G \log |f(x)| dx > -\infty$ then f has an inner-outer factorization if and only if $\log |f|$ is in $LRP(\Gamma_1)$.

A semigroup, P, in Γ_1 is called a half plane if $P \cup -P = \Gamma$ and $P \cap -P = \{0\}$. Helson and Lowdenslager [2] proved that if Γ_1 is a half plane then every function $f \in H^p(\Gamma_1)$ with $\int \log |f| dx > -\infty$ has a factorization as a product of an outer function, $h \in H^p(\Gamma_1)$ and an inner function, g, and this factorization is unique up to multiplication by constants of magnitude 1. From now on we shall assume $\int \log |f| dx > -\infty$.

Helson and Lowdenslager also showed [3] that if u is a real function such that u and e^u are summable, and v is the conjugate function of u with respect to the half plane, Γ_1, then e^{u+iv} is an outer function in $H^1(\Gamma_1)$. Conversely, if a summable outer function has the represention e^{u+iv} with u and v real then u is summable and v is equal to its conjugate modulo 2π except for an additive constant.

Let P be a half plane which contains Γ_1. Then, for $u \in L^1_R(G)$ there exists a conjugate function, v, which is unique if we assume $v(0) = 0$, such that $u + iv$ has its Fourier series supported on P. The function, v, is in L^p, $p < 1$. If u has its Fourier coefficients supported only on $\Gamma_1 \cup -\Gamma_1$ then $u + iv$ has its Fourier coefficients supported only on Γ_1 [4, Chap. 8, § 7]. Therefore, $f \in LRP(\Gamma_1)$ if