PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 2, 1977

ALGEBRAIC AUTOMORPHISMS OF ALGEBRAIC GROUPS WITH STABLE MAXIMAL TORI

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Let T_1 and T_2 be maximal tori of a connected linear algebraic group $G \subseteq GL(n, \kappa)$, and suppose some (algebraic group) automorphism σ of G stabilizes both T_1 and T_2. Suppose further that σ also stabilizes two Borel subgroups, B_1 and B_2, of G. This paper is about the following natural questions:

(1) Are T_1 and T_2 conjugate by a σ-fixed point of G?

(2) Are B_1 and B_2 conjugate by a σ-fixed point of G?

(3) If $T_i \subseteq B_i$, $(i = 1, 2)$, are the T_i and B_i respectively conjugate by a single σ-fixed point of G?

(4) Are at least T_1 and T_2 described in (3) above conjugate by a σ-fixed point of G?

In this paper is treated the case in which σ is an algebraic automorphism. If either $p = \text{char } \kappa = 0$ or σ is semisimple, then the answer to (4) above is yes; but there are counterexamples for (1), (2), and (3). (See below, Counterexamples A-1 and B.) Moreover, if both $p > 0$ and σ is not semisimple, then there is also a counterexample for question (4). (See below, Counterexample C.)

Incidental in the proofs is the simple result that when σ is algebraic, a σ-stable maximal torus is pointwise fixed by some finite power of σ, and by σ itself for $p = 0$, σ unipotent (Theorem 1).

Robert Steinberg has studied the questions above in [3], for the case that σ has finite fixed-point set in G, finding that the answers to questions (2), (3), and (4) are all yes. There is a counterexample for question (1) in the finite fixed-point set case, when the σ-stable maximal tori are not respectively contained in σ-stable Borel subgroups. (See below, Counterexample A-2.)

When σ is an algebraic automorphism of a general algebraic group G, its fixed-point set may be infinite. In fact, Steinberg shows (by [3], 10.10) that if σ is algebraic with finite fixed-point set, then G is necessarily solvable.

Throughout the paper the (now standard) terminology and basic results of Borel ([1] and [2]) are used, including the name *Borel subgroup* for a maximal solvable connected subgroup. In addition the mnemonic *clag* is used for a connected linear algebraic group, and the expression "the pair $T \subseteqq B$" for a maximal torus T and a Borel subgroup B containing T.

In all of the following theorems, G is a clag and σ an algebraic automorphism of G.