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SCHRÖDINGER AND DIRAC OPERATORS WITH SINGULAR POTENTIALS AND HYPERBOLIC EQUATIONS

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In an earlier paper we employed techniques from the theory of hyperbolic partial differential equations to deduce the self-adjointness of Dirac and Schrödinger operators with smooth potentials. The present paper applies these techniques to operators with singular potentials.

1. Introduction. There is a physical idea in back of our arguments which is especially clear in the case of operators of Dirac type. The Dirac operator $H = D + V$, a first-order partial differential operator, is the quantum Hamiltonian governing the dynamics of a relativistic particle in an external electromagnetic field. Intuitively, we expect that H (with domain C_0^∞) should be essentially self-adjoint if the time evolution is determined by the formal differential expression alone—that is, no boundary conditions are needed to tell the particle how to be scattered when it hits a singularity. First of all, we require that the underlying "physical space" be a complete Riemannian manifold so that no finite points are missing. Moreover, in a relativistic system waves propagate at the speed of light; hence compactly supported data are not propagated to infinity in a finite time, and thus no boundary conditions at infinity are required. Finally, suppose that the potential term V is locally well-behaved in a sense which we will make precise later—roughly that everywhere V is locally equal to a potential that requires no special boundary conditions. Then we expect that H is essentially self-adjoint. That is, by exploiting the finite propagation speed of the Dirac equation, we can patch together local good behavior to deduce global good behavior. This is the main result of §2.

In §§3 and 4 we apply analogous ideas to second-order Schrödinger operators, by considering the associated wave equations. Although the underlying ideas are similar to the Dirac case, there are a number of technical complications, some of which are dealt with in the preliminary material in §3. The conclusion is, roughly, that a Schrödinger operator with a locally well-behaved potential, which does not decrease too rapidly at infinity, is essentially self-adjoint; this global condition on the potential is needed because the nonrelativistic Schrödinger equation has infinite velocity of propagation.

(There is a large literature devoted to conditions which imply the essential self-adjointness of formally symmetric partial differential