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BETWEENNESS RELATIONS IN PROBABILISTIC METRIC SPACES

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Four distinct versions of the betweenness concept for probabilistic metric spaces are defined and studied. Conditions under which some or all of the properties of metric betweenness are satisfied are determined and the relationships among the different concepts are investigated.

1. Introduction. In his original paper on probabilistic metric spaces [8] K. Menger, in addition to introducing the basic concepts and axioms, introduced a definition of betweenness, developed some of its properties and showed that this relation was generally weaker than ordinary metric betweenness. Shortly thereafter, A. Wald [25] introduced a different definition of betweenness, based on a different triangle inequality, and showed that his relation did have all the properties of metric betweenness. Subsequently, J. F. C. Kingman [6] and F. Rhodes [16] studied betweenness in "Wald spaces" and H. Sherwood [23] considered a probabilistic version of the concept. Otherwise the subject has lain dormant—primarily because adequate tools for its analysis were not available. Our recent work on the structure of semigroups on the space of probability distribution functions [9, 11, 12, 17] and the development of "characteristic functions" for certain classes of these semigroups [10, 13, 14] has changed this state of affairs. Thus we return to the study of betweenness in probabilistic metric spaces. We focus our attention on four different versions of this concept. The first of these is the straightforward generalization of Wald's betweenness from Wald spaces to arbitrary probabilistic metric spaces. We show that this relation satisfies some, but generally not all, of the usual properties of metric betweenness, determine sufficient conditions for the validity of those properties which are not always satisfied and show that in some instances these conditions are also necessary. The second betweenness relation applies to a restricted but nevertheless very large class of probabilistic metric spaces. It always satisfies the metric betweenness properties and, whenever it is comparable to the first relation, it is either identical or weaker. The third relation, which applies to the same class of spaces as the second, is obtained by an extension of Kingman's idea from Wald spaces to this class. It is a metric betweenness for certain naturally defined metrics and is always weaker than the second relation. The last relation is Menger's betweenness. We reformulate Menger's definition in terms of triangle functions and show that in