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A NOTE ON MEASURES ON FOUNDATION SEMIGROUPS WITH WEAKLY COMPACT ORBITS

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For an extensive class of locally compact semigroups S, foundation semigroups with identity element, we prove that two subalgebras of $M(S)$ [the algebra of the bounded Radon measures on S] coincide. Namely, the algebra $L(S)$, generated by the $m \in M(S)^+$ for which the orbits on the compact subsets of S are weakly compact subsets of $M(S)$, or, equivalently, for which the translations are weakly continuous, and the algebra $M_e(S)$, generated by the $m \in M(S)^+$ for which the restrictions of the orbit of m on S to the compact subsets of S are weakly compact. In case S is a group, both these algebras consist of the bounded Radon measures that are absolutely continuous with respect to a Haar measure on S.

1. If S is a locally compact group and $m \in M(S)$ then the orbits F_m of m on all compact subsets F of S [$F_m := \{m * \bar{x} \mid x \in F\} \cup \{\bar{x} * m \mid x \in F\}$, where $\bar{x}$ denotes the point mass at x] are weakly compact subsets of $M(S)$ if and only if the restrictions $S_{m|F}$ of the orbit S_m of m on S to all compact subsets F of S [$S_{m|F} := \{\mu|_F \mid \mu \in S_m\}$] are weakly compact. The proof of this fact follows by observing that $F^{-1}K [:= \{x \in S \mid Fx \cap K \neq \emptyset\}]$ and KF^{-1} are compact as soon as both F and K are compact subsets of the group S. An arbitrary locally compact semigroup S may fail to have this compactness property and may [and actually does] give rise to two different subsets of $M(S)$: namely, to $L(S)$, the collection of all $m \in M(S)$ for which $F_{|m|}$ is weakly compact ($F \subseteq S$, compact), and to $M_e(S)$, the collection of all $m \in M(S)$ for which $S_{|m||F}$ is weakly compact ($F \subseteq S$, compact). Elementary properties of $L(S)$ can be found in e.g., [1], [2], [6], and [7] and of $M_e(S)$ in [4]. Although, in some respects $M_e(S)$ has better properties than $L(S)$ [cf. [4], e.g., (5.2) and (5.3)] $L(S)$ is a more obvious analogue of the group algebra than $M_e(S)$: $L(S)$ is a two sided L-ideal in $M(S)$ [cf. [1], (3, 4) and [2], (2.6)], while, in general, $M_e(S)$ is only an L-subalgebra of $M(S)$ [cf. (2.1) and [4], (2.6)].

It is natural to wonder about the relationship between $L(S)$ and $M_e(S)$. In view of the inner regularity of the measures in question, it is clear that $M_e(S) \subseteq L(S)$. As noted above, $M_e(S)$ may be strictly contained in $L(S)$, but for an important class of semigroups we can show that these collections coincide. We shall prove the following theorem.