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A CONSTRUCTIVE PROOF OF THE INFNIITE VERSION OF THE BELLUCE-KIRK THEOREM

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In [5], we proved the following infinite version of the Belluce-kirk theorem [1]:

THEOREM 1 [5]. *Let K be a nonempty weakly compact convex subset of a Banach space and assume that K possesses normal structure. Let F be a commutative family of nonexpansive self-mappings of K. Then $\mathscr{F}$ has a common fixed point.*

Fuchssteiner [3] recently proved an iteration theorem on partially ordered sets and derived several known fixed point theorems as consequences. This note is to respond to a final remark in [3]. We show that Theorem 1, indeed a more general one, can be proved without making use of the axiom of choice. We shall make use of the following theorem which can be proved constructively [2, Theorem I.2.5].

THEOREM 2 (*Zermelo* [7]). *Let $f: E \to E$ have the property that $f(x) \geqq x$ where $(E, \leqq)$ is a nonempty partially ordered set with the additional properties*:

(i) *If $a \leqq b$ and $b \leqq a$ then $a = b$;*

(ii) *Every chain in E has a least upper bound. Then f has a fixed point in E.*

Let (X, d) be a metric space and let $\{B_\alpha: \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of X, i.e., Λ is a directed set and $B_\alpha \subseteqq B_\beta$ if $\alpha \geqq \beta$. For each $x \in X$, let

$$r(x) = \lim_\alpha \sup\{d(x, y): y \in B_\alpha\} = \inf_\alpha \sup\{d(x, y) y \in B_\alpha\}$$

and

$$r = \inf\{r(x): x \in X\} .$$

The set $\{x \in X: \mathrm{r}(x) = r\}$ (the number r) will be called the asymptotic center (asymptotic radius) of $\{B_\alpha: \alpha \in \Lambda\}$ w.r.t. X. For a set C in a topological space, cl(C) will denote its closure. A topological semigroup S is said to be left reversible if any two nonempty closed right ideals of S have a nonvoid intersection (cf. [4]). An action of a topological semigroup S on X is a mapping ψ from $S \times X$ into X denoted by $\psi(s, x) = s(x)$ such that $(s_1 s_2)(x) = s_1(s_2(x))$ for all $s_1, s_2 \in S$,