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DEDEKIND'S PROBLEM: MONOTONE BOOLEAN FUNCTIONS ON THE LATTICE OF DIVISORS OF AN INTEGER

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This paper is concerned with the combinatorial problem of counting the number of distinct collections of divisors of an integer N having the property that no divisor in a collection is a multiple of any other. It is shown that if N factors into primes $N=p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n}$ the number of distinct collections of divisors with the stated property does not exceed $(\sum_{i=1}^n a_i-n+3)^M$, where M is the maximum coefficient in the expansion of the polynomial

$$(1+x+x^2+\cdots+x^{a_1})(1+x+x^2+\cdots+x^{a_2})\cdots(1+x+x^2+\cdots+x^{a_n}) .$$

In the special case where N is squarefree the problem is equivalent to that of counting the number of "Sperner families" on n letters, for which G. Hansel obtained the upper bound 3^{M_n}, where M_n is the binomial coefficient $\binom{n}{[n/2]}$; the result in this paper is then a generalization of Hansel's theorem to the non-squarefree case.

The problem has also been formulated as that of counting the number of families consisting of incomparable subsets of a set of n objects (the objects of course corresponding to the primes in the number-theoretic formulation), with the variation that each object may appear in a set with a specifically limited number of repetitions (these limits corresponding to the prime exponents).

NOTATION. Given n letters $x_1, x_2, \cdots, x_n$, and n positive integers $a_1, a_2, \cdots, a_n$, consider the lattice consisting of all terms $(x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n})$ in the polynomial $\prod_{i=1}^n(\sum_{k=0}^{a_i} x_i^k)$, with the partial ordering defined $(x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}) \subseteqq (x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n})$ if $j_i \leqq k_i$ for all i. A single term $X = (x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n})$ in this lattice will be referred to as a "set", the empty set ϕ denoting the term with all exponents $j_1, j_2, \cdots, j_n$ equal to zero. If $X = (x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n})$, the notation (X, x_k^c) will indicate the set $(x_1^{j_1}x_2^{j_2}, \cdots x_k^{j_k+c}\cdots x_n^{j_n})$, and the exponent sum $j_1 + j_2 + \cdots + j_n$ will be written $|X|$.

A monotone Boolean function is defined to be a function taking the values 0 or 1 on each set of this lattice with the property that $f(X) \leqq f(Y)$ if $X \subseteqq Y$. The problem of counting the number of monotone Boolean functions on this lattice is then equivalent to the problem concerning collections of divisors of N stated at the begin-