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# DISCRETE GENERALIZED GRONWALL INEQUALITIES IN THREE INDEPENDENT VARIABLES

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**The objective of this paper is to establish some new discrete inequalities of the Gronwall type in three independent variables which can be used in the analysis of a class of finite difference equations involving three independent variables.**

1. Introduction. The role played by the discrete inequalities of the Gronwall type [3] in the theory of finite difference equations and numerical analysis is well known (see, [4]-[8] and the references therein). Recently, in a series of papers [4]-[8], Pachpatte has established a number of new discrete inequalities of the Gronwall type which can be used in the theory of discrete time systems involving one independent variable. To our knowledge such inequalities have not been considered before and seem to have much future in the literature.

2. Main results. Before giving the main results in this section, we first recollect a few of the basic notions and definitions from [4]-[8]. Let $N_0 = \{0, 1, 2, \cdots\}$. The expression $u(0) + \sum_{s=0}^{n-1} b(s)$ represents a solution of the linear difference equation $\Delta u(n) = b(n)$ for all $n \in N_0$, where $\Delta$ is the operator defined by $\Delta u(n) = u(n+1) - u(n)$. The expression $u(0) \prod_{s=0}^{n-1} c(s)$ represents a solution of the linear difference equation $u(n+1) = c(n)u(n)$ for all $n \in N_0$. We use the usual convention of writing $\sum_{s \in \Phi} b(s) = 0$ and $\prod_{s \in \Phi} c(s) = 1$, if $\Phi$ is the empty set. We also use the following notions of the operators $\Delta u_x(x, y, z) = u(x+1, y, z) - u(x, y, z)$, $\Delta u_y(x, y, z) = u(x, y+1, z) - u(x, y, z)$, $\Delta u_z(x, y, z) = u(x, y, z+1) - u(x, y, z)$ and $\Delta u_{xy}^2(x, y, z) = \Delta u_x(x, y+1, z) - \Delta u_x(x, y, z)$ and so on. We often use the letters $x$, $y$, and $z$ to denote the three independent variables which are the members of $N_0$. For $x, y, z \in N_0$, and functions $a$, $b$, $c$ with domain $N_0$, and $p$ with domain $N_0^3$, set

$$\phi(x, y, z; a, b, c; p) = [a(0) + b(y) + c(z)] \prod_{s=0}^{x-1} \left[ 1 + \frac{\Delta a(s)}{a(s) + b(0) + c(z)} + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s, t, r) \right]. \tag{A}$$

A useful three independent variable discrete inequality is embodied in the following theorem.