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SOME RELATIONSHIPS BETWEEN MEASURES

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Suppose μ and ν are (nonnegative, countably additive) measures on the same sigma-ring. We say that ν is *quasi-dominant* with respect to μ if each measurable set contains a subset with the same ν-measure, where μ is absolutely continuous with respect to ν on that subset. In particular, ν is quasi-dominant with respect to μ if μ is sigma-finite. We say that ν is *strongly recessive* with respect to μ if the zero measure is the only measure that is quasi-dominant with respect to μ and less than or equal to ν. Properties of these relationships are investigated, and applications are given to purely atomic measures, to the Radon-Nikodým theorem and to a decomposition of product measures.

1. Weak singularity and absolute continuity. Let μ and ν be (nonnegative, countably additive) measures on a sigma-ring $\mathscr{S}$. Recall that ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$, if $\nu(E) = 0$ whenever $\mu(E) = 0$. If $\nu \ll \mu$ and $\mu \ll \nu$, then μ and ν are said to be equivalent and we write $\mu \sim \nu$. We say that ν is *weakly singular* with respect to μ, denoted $\nu S \mu$, if given E in $\mathscr{S}$, there exists F in $\mathscr{S}$ such that $\nu(E) = \nu(E \cap F)$ and $\mu(F) = 0$.

We shall make use of the following form of the Lebesgue Decomposition Theorem [3, Theorem 2.1 or 6, Theorem 1.1]:

THEOREM 1.1. (*Lebesgue Decomposition Theorem*). *Suppose μ and ν are measures on a sigma-ring $\mathscr{S}$. Then there exist measures ν_1 and ν_2 such that* (1) $\nu = \nu_1 + \nu_2$, (2) $\nu_1 \ll \mu$ *and* (3) $\nu_2 S \mu$. *The measure ν_2 is unique. We may arrange to have $\nu_1 S \nu_2$, and under that requirement ν_1 is unique also.*

If ν is a measure on $\mathscr{S}$ and $A \in \mathscr{S}$, let ν_A be the measure given by $\nu_A(E) = \nu(A \cap E)$ for all $E \in \mathscr{S}$.

THEOREM 1.2. *Suppose $M_1(\mathscr{S})$ and $M_2(\mathscr{S})$ are families of measures on $\mathscr{S}$ such that the zero measure is the only measure common to both families and such that ν_A is in one of the families whenever ν is in that family and $A \in \mathscr{S}$. Suppose, moreover, that each measure ν on $\mathscr{S}$ can be written as the sum of measures ν_1 and ν_2 such that $\nu_1 \in M_1(\mathscr{S})$ and $\nu_2 \in M_2(\mathscr{S})$ and $\nu_1 S \nu_2$. Then $\nu \in M_2(\mathscr{S})$ if and only if $\nu(A) = 0$ whenever $\nu_A \in M_1(\mathscr{S})$.*

Proof. Suppose $\nu \in M_2(\mathscr{S})$. Then $\nu_A \in M_2(\mathscr{S})$ for all $A \in \mathscr{S}$. If