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MEASURES AS FUNCTIONALS ON UNIFORMLY CONTINUOUS FUNCTIONS

JAN K. PACHL

The space $\mathfrak{M}_t$ of bounded Radon measures on a complete metric space is studied in duality with the space $\mathscr{U}_b$ of bounded uniformly continuous functions. The weak topology has reasonable properties: the space $\mathfrak{M}_t$ is $\mathscr{U}_b$-weakly sequentially complete, and every $\mathscr{U}_b$-weakly compact subset of $\mathfrak{M}_t$ is pointwise equicontinuous on the set of 1-Lipschitz functions.

1. Introduction. Let (X, d) be a complete metric space and $\mathfrak{M}_t(X)$ the space of (bounded) Radon (=tight) measures on X. This space is usually studied in duality with the space $\mathscr{C}_b(X)$ of bounded continuous functions on X. It is known that the weak topology $w(\mathfrak{M}_t(X), \mathscr{C}_b(X))$ is sequentially complete, and there is a useful criterion (Prohorov's condition) for $w(\mathfrak{M}_t, \mathscr{C}_b)$-compactness [11].

In this paper we turn to the space $\mathscr{U}_b(X)$ of bounded uniformly continuous functions on X and to the weak topology $w(\mathfrak{M}_t(X), \mathscr{U}_b(X))$. The topologies $w(\mathfrak{M}_t, \mathscr{C}_b)$ and $w(\mathfrak{M}_t, \mathscr{U}_b)$ coincide on the positive cone $\mathfrak{M}_t^+$; thus our results say nothing new about positive measures. Obviously, the two topologies differ (on $\mathfrak{M}_t$) whenever $\mathscr{U}_b \neq \mathscr{C}_b$.

The main results are: (A) the topology $w(\mathfrak{M}_t, \mathscr{U}_b)$ is sequentially complete, and (B) a norm-bounded subset of $\mathfrak{M}_t$ is relatively $w(\mathfrak{M}_t, \mathscr{U}_b)$-compact if and only if its restriction to the set

$$\mathrm{Lip}\,(1) = \{f\colon X \to R \mid \|f\| \leqq 1 \text{ and } |f(x) - f(y)| \leqq d(x, y) \text{ for } x, y \in X\}$$

is equicontinuous in the compact-open topology.

The topology of uniform convergence on Lip (1) was discussed by Dudley [3]. Here we improve some of Dudley's results. For example, Theorem 6 in [3] says, in the present setup, that $\mu_n \to \mu$ uniformly on Lip (1) whenever $\mu \in \mathfrak{M}_t$, $\mu_n \in \mathfrak{M}_t$ for $n = 1, 2, \cdots$, and $\mu_n(f) \to \mu(f)$ for each $f \in \mathscr{C}_b(X)$. Here we obtain the same conclusion, assuming only that $\mu_n(f) \to \mu(f)$ for each $f \in \mathscr{U}_b(X)$.

A reasonable generalization is to allow X to be an arbitrary uniform space and replace $\mathfrak{M}_t$ by the space $\mathfrak{M}_u(X)$ of uniform measures on X (see [4] and the references therein). The results extend to the space $\mathfrak{M}_u(X)$, as well as to the space $\mathfrak{M}_F(X)$ of free uniform measures. Several previously studied spaces of measures can be described as $\mathfrak{M}_u$ or $\mathfrak{M}_F$—see [5], [8]. To cover both $\mathfrak{M}_u$ and $\mathfrak{M}_F$, in § 2 we employ sets of Lipschitz functions more general than Lip (1).

As in similar situations studied before (e.g., [1], [10]), the goal