PACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 1, 1979

THE HYPERBOLIC METRIC AND COVERINGS OF RIEMANN SURFACES

CARL DAVID MINDA

Suppose X and Y are Riemann surfaces which have the open unit ball as universal covering surface. Let $d\sigma_X$, $d\sigma_Y$ be the hyperbolic metric on X, Y, respectively. Given any analytic function $f: X \to Y$ the principle of hyperbolic metric asserts that $(f^*(d\sigma_Y)/d\sigma_X)(p) \leqq 1$ for each point $p \in X$ where $f^*(d\sigma_Y)$ is the pull-back to X via f of the hyperbolic metric on Y. Moreover, equality holds if and only if f is an (unbranched, unlimited) covering of X onto Y. This paper has two main objectives. The first is to study how the principle of hyperbolic metric can be strengthened if we only consider analytic functions which are not coverings. The second is to investigate the set of all analytic coverings of X onto Y.

1. Notation and terminology. Throughout this paper, unless the contrary is explicitly stated, X and Y will always denote Riemann surfaces whose universal covering surface is the open unit ball $\boldsymbol{B}$. The set of all analytic functions $f: X \to Y$ will be denoted by $A(X, Y)$. Often we will fix points $p \in X, q \in Y$ and consider analytic functions $f: (X, p) \to (Y, q)$. This notion implies that $f(p) = q$.

We shall make free use of the theory of covering surfaces. For example, the material in [3, pp. 27-44] or [15, Ch. 5] is sufficient for our purposes. To say that $f: X \to Y$ is an analytic covering projection will always mean that X is an (unbranched, unlimited) covering surface of Y and f is an analytic function. Let $C(X, Y)$ denote the set of all analytic coverings $f: X \to Y$; it is possible that $C(X, Y)$ is empty. $N(X, Y)$ is the complement of $C(X, Y)$ in $A(X, Y)$. One basic result we shall need is the following. Suppose $f: X \to Y$, $g: Y \to Z$, and $h: X \to Z$ are analytic mappings of Riemann surfaces such that $g \circ f = h$. Then if any two of these functions are coverings, so is the third. In particular, the composition of coverings is again a covering.

Given a Riemann surface X with the unit ball as universal covering surface, there is a unique conformal metric $d\sigma_X = \lambda_X(z)|dz|$ of constant curvature -4 on X called the hyperbolic metric. It is defined on X by projecting the Poincaré hyperbolic metric on $\boldsymbol{B}$ onto X by means of any analytic universal covering projection. For any analytic function $f: X \to Y$ we will let $f^*(d\sigma_Y)$ denote the pull-back to X via the function f of the hyperbolic metric on Y. Note that if $\pi: \boldsymbol{B} \to X$ is an analytic universal covering projection, then $\pi^*(d\sigma_X) = d\sigma_B$. If $ds_1 = \rho_1(z)|dz|$ and $ds_2 = \rho_2(z)|dz|$ are two conformal