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A COMMUTATIVE BANACH ALGEBRA OF FUNCTIONS OF GENERALIZED VARIATION

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It is known that the space of functions, anchored at a, and having bounded variation form a commutative Banach algebra under the total variation norm. We show that functions of bounded kth variation also form a Banach algebra under a norm defined in terms of the total kth variation.

1. Introduction. Let $BV_1[a, b]$ denote the space of functions of bounded variation on the closed interval $[a, b]$, and denote the total variation of f on that interval by $V_1(f)$ or $V_1(f; a, b)$. If

$$BV_1^*[a, b] = \{f;\ V_1(f) < \infty,\ f(a) = 0\}\ ,$$

then it is a well known result that $BV_1^*[a, b]$ is a Banach space under the norm $\|\cdot\|_1$, where $\|f\|_1 = V_1(f)$. What appears to be less well known is that, using pointwise operations, $BV_1^*[a, b]$ is a commutative Banach algera with a unit under $\|\cdot\|_1$ — see for example [1] and Exercise 17.35 of [2].

In [4] it was shown that $BV_k[a, b]$ is a Banach space under the norm, $\|\cdot\|_k$, where

$$\|f\|_k = \sum_{s=0}^{k-1} |f^{(s)}(a)| + V_k(f; a, b)\ , \tag{1}$$

and where the definition of $V_k(f; a, b) \equiv V_k(f)$ can be found in [3]. The subspace

$$BV_k^*[a, b] = \{f;\ f \in BV_k[a, b],\ f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0\}$$

is clearly also a Banach space under the norm $\|\cdot\|_k^*$, where

$$\|f\|_k^* = \alpha_k V_k(f)\ , \tag{2}$$

and $\alpha_k = 2^{k-1}(b - a)^{k-1}(k - 1)!$.

If we define the product of two functions in $BV_k^*[a, b]$ by pointwise multiplication, then we show, in addition, that $BV_k^*[a, b]$ is a commutative Banach algebra under the norm given in (2). It is obvious that $BV_k^*[a, b]$ is commutative, so our main programme now is to show that if f and g belong to $BV_k^*[a, b]$, then so does fg, and that

$$V_k(fg) \leqq 2^{k-1}(k - 1)!\,(b - a)^{k-1} V_k(f) V_k(g)\ , \qquad k \geqq 1\ .$$