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NONLINEAR SMOOTH REPRESENTATIONS OF COMPACT LIE GROUPS

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We study semi-free (=free off the fixed-point set) smooth actions of a compact Lie group G on disks and spheres with fixed-point set a disk or sphere, respectively. In dimensions $\geqq 6$ and codimension $\neq 2$ we obtain a complete classification for such actions on disks and a partial classification for spheres, together with partial results in dimension 5 or codimension 2. We show that semi-free smooth actions of G on the n-disk D^n, $n \geqq 6 + \dim G$, with fixed-point set an $(n$-$k)$-disk, $k \neq 2$, are classified by two invariants:

(1) a free orthogonal action of G on the $(k$-$1)$-sphere S^{k-1} (the representation at the fixed points) and

(2) an element of the Whitehead group $\mathrm{Wh}(\pi_0(G))$.

In fact (§ 4, Theorem A), there is a bijection τ from the set $\mathscr{D}_\rho^{n,k}$ of such actions, with a given representation $\rho\colon G \to O(k)$ at the fixed points, onto $\mathrm{Wh}(\pi_0(G))$, and for $n - k \geqq 2$, $\mathscr{D}_\rho^{n,k}$ is a group under equivariant boundary connected sum and τ is an isomorphism. The corresponding set $\mathscr{S}_\rho^{n,k}$ of actions on spheres also forms a group (§ 4, Corollary 4).

For $G = Z_m = Z/mZ$ we show (§4, Corollary 5) that these actions on D^n restrict to distinct actions on $\partial D^n = S^{n-1}$ if $n - 1$ is odd. For $n = k$ these actions on S^{n-1} are free (since $S^{n-k-1} = S^{-1} = \varnothing$) and, in fact, are the same as those constructed by Milnor in [20], where he used Reidemeister torsion to distinguish infinitely many of them. We observe that his later application [21, Corollary 12.13] of the Atiyah-Bott fixed-point formula [1, § 7] implies they are all distinct. For $n > k$ we use Whitehead torsion to distinguish them, employing the result of [11] and [3, Prop. 4.14] that $\mathrm{Wh}(Z_m)$ is free abelian. (For analogous applications of Reidemeister torsion *versus* Whitehead torsion com- pare [19] vs. [37] and [33, 36] vs. [31].)

Thus for $m \neq 1, 2, 3, 4$ or 6, which according to [11] implies $\operatorname{rank} \mathrm{Wh}(Z_m) > 0$, we obtain (§ 4, Corollary 6) infinitely many different semi-free smooth actions of Z_m on every sphere of odd dimension greater than four, with fixed-point set a sphere of any even codimension at least four. These actions are not smoothly equivalent to linear actions, although they are topologically linear according to [38] and [9].

Invariant (1) of a semi-free action is equivalent to a representation $\rho\colon G \to O(k)$ that is "fixed-point-free", i.e., such that $\rho(g)$ has no eigenvalue equal to $+1$ for $g \in G$, $g \neq$ identity. The only G which