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NOTES ON GENERALIZED BOUNDARY VALUE PROBLEMS IN BANACH SPACES, I ADJOINT AND EXTENSION THEORY

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Let A: $X \to Y$ be a densely defined closed operator where X and Y are Banach spaces. Let F be a locally convex topological vector space and H: $X \to F$ an operator such that $N(H)$ and $D(A)$ have nontrivial intersection and $D(H^*)$ is total over F. We compute A_H^* and A_H^* where A_H is the operator determined by A on $N(H)$ and $A_H(x) = (Ax, Hx)^t$.

We also characterize certain closed extensions of A_H and the adjoints of these extensions. In particular application is made to the problem of determining self-adjoint extensions of symmetric operators restricted by boundary conditions in a Hilbert space.

1. Introduction. Suppose X, Y and A are as above. Let H be an operator having domain in X and range in a locally convex topological vector space (l.c.t.v.s.) F. Assume that $D(A) \cap N(H)$ is nontrivial. Then the system

$$\begin{aligned} Ax &= f \\ Hx &= r \end{aligned} \tag{1.1}$$

is called a generalized boundary value problem (b.v.p). We call the first equation of (1.1) the *operator part* of the b.v.p. and the second the *boundary condition*. H is the *boundary operator*. If $r = 0$ the problem is said to be homogeneous, otherwise it is nonhomogeneous. In the nonhomogeneous case, (1.1) determines an operator $\mathscr{A}_H$: $X \to Y \times F$ and in the homogeneous case an operator $A_H \subset A$: $X \to Y$ on

$$D(A_H) := \{x \in D(A): Hx = 0\} .$$

In this paper we are going to construct the adjoints A_H^* and $\mathscr{A}_H{}^*$ and compare their structure. Knowledge of A_H^* and $\mathscr{A}_H{}^*$ yield at once statements of Fredholm alternative solvability conditions for the original b.v.p. We will also be interested in the following extension problem. Suppose A and B: $Y^* \to X^*$ are 1-1 and $B^* \supset A$. Let K: $Y^* \to G$ (G a l.c.t.v.s) be a boundary operator. Then (roughly speaking)

$$A_H \subset A \subset B_K^* . \tag{1.2}$$

One can now ask for the structure of all closed extensions of A_H