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COMPACT AND WEAKLY COMPACT DERIVATIONS OF C^*-ALGEBRAS

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In a forthcoming paper, the second-named author asks if every compact derivation of a C^*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-module X is the uniform limit of finite-rank derivations. We answer this question affirmatively in the present paper when $X = \mathscr{A}$ by characterizing the structure of compact derivations of C^*-algebras. In addition, the structure of weakly compact derivations of C^*-algebras is determined, and as immediate corollaries of these results, necessary and sufficient conditions are given for a C^*-algebra to admit a nonzero compact or weakly compact derivation.

To fix our notation, we we recall some basic definitions. A derivation of a C^*-algebra $\mathscr{A}$ is a linear map $\delta: \mathscr{A} \to \mathscr{A}$ for which $\delta(ab) = a\delta(b) + \delta(a)b$, $a, b \in \mathscr{A}$. If $x \in \mathscr{A}$, the map $a \to ax - xa$, $a \in \mathscr{A}$, defines a derivation of $\mathscr{A}$ which we denote by adx.

By an ideal of a C^*-algebra, we always mean a uniformly closed, two-sided ideal.

A C^*-algebra $\mathscr{A}$ is said to *act atomically* on a Hilbert space H if there exists an orthogonal family $\{P_\alpha\}$ of projections in $B(H)$, each commuting with $\mathscr{A}$, such that $\bigoplus_\alpha P_\alpha$ is the identity operator on H, $\mathscr{A}P_\alpha$ acts irreducibly on $P_\alpha(H)$, and $\mathscr{A}P_\alpha$ is not unitarily equivalent to $\mathscr{A}P_\beta$ for $\alpha \neq \beta$.

If $\{\mathscr{A}_n\}$ is a sequence of C^*-algebras, $\bigoplus_n \mathscr{A}_n$ denotes the C^*-direct sum of the $\mathscr{A}_n$'s, i.e., $\bigoplus_n \mathscr{A}_n$ is the C^*-algebra of all uniformly bounded sequences $\{a_n\}$, $a_n \in \mathscr{A}_n$, equipped with pointwise operations and the norm $\|\{a_n\}\| = \sup_n \|a_n\|$. $\hat{\bigoplus}_n \mathscr{A}_n$ denotes the C^*-subalgebra of $\bigoplus_n \mathscr{A}_n$ formed by all sequences $\{a_n\}$ with $\|a_n\| \to 0$.

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2. Compact derivations. The following lemma is due to Ho ([3], Corollary 1):

LEMMA 2.1. *Let H denote an infinite dimensional Hilbert space, $B(H)$ the algebra of all bounded linear operators on H. If δ is a compact derivations of $B(H)$, then $\delta \equiv 0$.*