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SUPPORT POINT FUNCTIONS AND THE LOEWNER VARIATION

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1. Introduction. Let $U = \{z: |z| < 1\}$ and $\mathscr{S}$ the set of functions $f, f(z) = z + a_2 z^2 + \cdots$, that are analytic and 1:1 in U. Denote by σ the collection of support point functions of $\mathscr{S}$, i.e., functions $f \in \mathscr{S}$ that satisfy

$$\operatorname{Re} L(f) = \max_{g \in \mathscr{S}} \operatorname{Re} L(g)$$

for some nonconstant continuous (in the topology of local uniform convergence) linear functional on $\mathscr{S}$. Finally, denote by $E(\mathscr{S})$ the set of extreme point functions of $\mathscr{S}$.

It is well known that if $f \in \sigma \cup E(\mathscr{S})$, then $f(U)$ is the complement of a single Jordan arc extending from some finite point to ∞ and along which $|w|$ is strictly increasing. Indeed, this has been demonstrated for the class $E(\mathscr{S})$ by L. Brickman [1] and for the class σ by A. Pfluger [5] (see also L. Brickman and D. Wilken [2]). Consequently, if $f \in \sigma \cup E(\mathscr{S})$, there is a Loewner chain

$$f(z, t) = e^t\left[z + \sum_{n=2}^{\infty} a_n(t) z^n\right] \quad (0 \leqq t < \infty)$$

with $f(z, 0) = f(z)$ and $f(z, t_1)$ subordinate to $f(z, t_2)$ if $0 \leqq t_1 < t_2 < \infty$ (see [6, p. 157]). Note that $e^{-t} f(z, t) \in \mathscr{S}$. Let $w(z, t) = e^{-t}(z + \hat{b}_2(t) z^2 + \hat{b}_3(t) z^3 + \cdots)$ be analytic for $t \in \{t: 0 \leqq t < \infty\}$ and $z \in U$, 1:1 in U with $|w(z, t)| < 1$, and such that $f(z) = f(w(z, t), t)$ for each $t \in \{t: 0 \leqq t < \infty\}$ and all $z \in U$. Observe that we define $\hat{w}(z, t) \equiv e^t w(z, t) = z + \hat{b}_2(t) z^2 + \cdots \in \mathscr{S}$ and that $|\hat{w}(z, t)| < e^t$ for $z \in U$.

In §2 it is shown that if $f \in E(\mathscr{S})$, then $e^{-t} f(z, t) \in E(\mathscr{S})$ and also that if $f \in \sigma$, then $e^{-t} f(z, t) \in \sigma$. This latter result is a generalization of a theorem due to S. Friedland and M. Schiffer [3, p. 143]. Also, in the process of generalizing this theorem a fairly easy method is established for finding for each t, $0 \leqq t < \infty$, a continuous linear functional which $e^{-t} f(z, t)$ maximizes.

2. Preservation of the sets σ and $E(\mathscr{S})$ under the Loewner variation. It is easy to show that if $f \in E(\mathscr{S})$, then $e^{-t} f(z, t) \in E(\mathscr{S})$ also. Indeed, if this were not the case, then there would exist distinct functions $f_1, f_2 \in \mathscr{S}$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ for which $\lambda_1 f_1(z) + \lambda_2 f_2(z) = e^{-t} f(z, t)$. This would imply that $e^t \lambda_1 f_1(w(z, t)) + e^t \lambda_2 f_2(w(z, t)) = f(w(z, t), t) = f(z)$. Since $e^t f_1(w(z, t))$ and $e^t f_2(w(z,t))$ are in $\mathscr{S}$, the fact that $f(z) \in E(\mathscr{S})$ is contradicted and therefore